Review Sheet

Saturday, February 29, 2020 4:19 PM

First Order ODE's $(y' = F(x,y))$

1 Seperable

X: generic independent variable

Y: generic dependent variable

$$\frac{dy}{dx} = f(x) \cdot g(y) \longrightarrow \int \frac{dy}{g(y)} = \int f(x)\,dx + C$$

$$G(x,y) = C$$ } Depends on initial conditions

"Family of Curves"

2 Homogenous

A function of the form $F(x,y)$ is considered homogenous of order n if $F(tx, ty) = t^n F(x,y)$

If $n=0$, $F(tx, ty) = F(x,y)$

Set $z = \frac{y}{x}$ $\therefore y = z \cdot x$

Treat x as t (only works when $n=0$)

$$\frac{d}{dx}(x \cdot z) = F(x, xz) = F(1, z)$$

Product Rule $x\frac{dz}{dx} + z = F(1, z)$

$$x\frac{dz}{dx} = F(1,z) - z$$ } Seperable ODE

$$\frac{dz}{F(1,z)-z} = \frac{dx}{x}$$

* Review Antenna Example

3 Linear

$$a_1(x)\frac{dy}{dx} + a_0(x)\,y = b(x) \quad \text{where } a_1 \neq 0$$

1. Divide by a_1 → "Standard Form"

$$\frac{dy}{dx} + P(x)\,y = Q(x) \qquad \begin{cases} P(x) = \dfrac{a_0(x)}{a_1(x)}, \\ Q(x) = \dfrac{b(x)}{a_1(x)} \end{cases}$$

2. Find integrating factor, μ

$$\mu_x = e^{\int P(x)\,dx}$$

3. Use Formula

$$y(x) = \frac{1}{\mu_x}\left(\int \mu_x Q(x)\,dx + C\right)$$

4 Bernoulli

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n \quad \text{(if } n=0\text{, linear} \uparrow\text{)}$$

1. Get in standard form

$$P(x) = \frac{a_0(x)}{a_1(x)} \qquad Q(x) = \frac{b(x)}{a_1(x)}$$

2. Does $n=1$?

If $n=1$ $\quad y(x) = \overset{\text{a constant}}{\overset{\downarrow}{A}}\, e^{\int Q(x)-P(x)\,dx}$

3. Otherwise find μ

$$\mu(x) = e^{\int (1-n)P(x)\,dx}$$

4. Use Formula

$$y(x) = \left[\frac{1}{\mu(x)}\left(\int (1-n)\mu(x)\,Q(x)\,dx + C\right)\right]^{\frac{1}{1-n}}$$

5 Exact

$$\frac{dy}{dx} = F(x,y) \overset{\text{can be written as}}{\Longrightarrow} M(x,y)\,dx + N(x,y)\,dy = 0$$

If $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ the equation is exact $\left(M_y = N_x\right)$

1. Integrate M w/ respect to x, w/ a potential constant function of y

$$f(x,y) = \int_x M\,dx + G(y)$$

2. Solve for $G(y)$

$$\frac{\partial f}{\partial y} = N = \frac{\partial}{\partial y}\left(\int_x M\,dx + G(y)\right)$$

Special Case? Almost Exact But Not Quite

If $\dfrac{M_y - N_x}{N} = g(x)$ or $\dfrac{N_x - M_y}{M} = g(y)$

Function of only x Function of only y

1. Use Integrating Factor

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}\,dx} \quad \text{or} \quad \mu(y) = e^{\int \frac{N_x - M_y}{M}\,dy}$$

2. Follow steps outlined above but with $M \cdot \mu$

2. Follow steps outlined above but with [illegible]

Solving 2nd Order ODE's ($y'' = F(x, y, y')$)

1. Convert 2nd Order to Systems of First Order

A. Dependent Variable is Missing (No y)

Set $v = y' \therefore v' = y''$

$$y'' = F(x, y') \xrightarrow{\text{convert}} v' = F(x, v)$$

Solve for v, then $v = y' \therefore y = \int v$

B. Independent Variable is Missing

$$y'' = F(y, y') \qquad v = \frac{dy}{dx} = y' \therefore v' = y''$$

$$v' = F(y, v) \qquad v = \frac{dy}{dx} = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy} = \frac{1}{2}\frac{d\,v^2}{dy}$$

Use this to form a separable ode and solve for v

$v = y' \therefore y = \int v$

2. Linear

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$$

convert to standard form

$$y'' + P(x)y' + Q(x)y = R(x) \quad \begin{cases} P(x) = \dfrac{a_1(x)}{a_2(x)} \\ Q(x) = \dfrac{a_0(x)}{a_2(x)} \\ R(x) = \dfrac{b(x)}{a_2(x)} \end{cases}$$

If $y(x)=0$ is a valid solution or $b(x)=0$ the Equation is Homogenous

General Homogenous Solution:

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x) \quad \begin{cases} c_1, c_2 \text{ are arbitrary constants} \\ y_1, y_2 \text{ solutions s.t. Wronskian of } y_1, y_2 \neq 0 \end{cases}$$

Non Homogenous?

General Solution $\Rightarrow$ $y = \underbrace{y_h}_{\text{General homogenous solution}} + \underbrace{y_p}_{\text{Particular solution}}$

Tips for finding y_p

If $\frac{b(x)}{a_0(x)} = \text{constant}$ Then $p(x) = \frac{b(x)}{a_0(x)}$

If not, use Green's Theorem (Discussed Later #8)

3 The Wronskian

$n \times n$ determinant

$$W[y_1, y_2, \cdots, y_n] = \begin{vmatrix} y_1(x) & \cdots & y_n(x) \\ y_1'(x) & \ddots & \vdots \\ \vdots & & \vdots \\ y_1^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}$$ is not 0 for all $\alpha < x < \beta$

4 Constant Coefficient

$ay'' + by' + cy = 0$ $\{a, b, c$ are all constants

1. Try to factor

$ar^2 + br + c = 0$

$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

2. Use table

$D = b^2 - 4ac$

D	$y_1(x)$	$y_2(x)$	Info
$D<0$	$e^{\alpha x}\cos(\beta x)$	$e^{\alpha x}\sin(\beta x)$	$\alpha = \frac{-b}{2a}$ $\beta = \frac{\sqrt{-D}}{2a}$
$D=0$	$e^{\alpha x}$	$x e^{\alpha x}$	$\alpha = \frac{-b}{2a}$
$D>0$	$e^{\alpha x}\cosh(\gamma x)$ $e^{(\alpha-\gamma)x}$	$e^{\alpha x}\sinh(\gamma x)$ $e^{(\alpha+\gamma)x}$	$\alpha = \frac{-b}{2a}$ $\gamma = \sqrt{\frac{D}{2a}}$

5 Equidimensional (Cauchy-Euler)

$ax^2 y'' + bxy' + cy = 0$ $\{a, b, c$ are all constants

$D = (b-a)^2 - 4ac$

D	$y_1(x)$	$y_2(x)$	Info
$D<0$	$\lvert x\rvert^{\alpha}\cos(\beta \ln\lvert x\rvert)$	$\lvert x\rvert^{\alpha}\sin(\beta \ln\lvert x\rvert)$	$\alpha = \frac{-(b-a)}{2a}$ $\beta = \frac{\sqrt{-D}}{2a}$
$D=0$	$\lvert x\rvert^{\alpha}$	$\lvert x\rvert^{\alpha}\ln\lvert x\rvert$	$\alpha = \frac{-(b-a)}{2a}$
$D>0$	$\lvert x\rvert^{\alpha}\cosh(\gamma \ln\lvert x\rvert)$ $\lvert x\rvert^{\alpha+\gamma}$	$\lvert x\rvert^{\alpha}\sinh(\gamma \ln\lvert x\rvert)$ $\lvert x\rvert^{\alpha-\gamma}$	$\alpha = \frac{-(b-a)}{2a}$ $\gamma = \frac{\sqrt{D}}{2a}$

6 Guessing e^x

For $a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$

If $a_2(x) + a_1(x) + a_0(x) = 0$ $\}$ We know a solution to the ODE is e^x

$y_1(x) = e^x$

If $a_2(x)+a_1(x)+a_0(x)=0$ } We know a solution to the ODE is e^x

$y_1(x)=e^x$

7 Abel's Equation (Finding 2nd Solution)

Given a solution $y_1(x)$, a second linearly independent solution $y_2(x)$ can be constructed using:

$$y_2(x) = A\cdot y_1(x)\int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx \quad \text{where } P(x)=\frac{a_1(x)}{a_2(x)}$$

8 Finding A Particular Solution w/ Green's Theorem

For a nonhomogenous equation, a particular solution can be constructed w/ this formula if the homogenous solution is known ($y_1(x)$, $y_2(x)$)

$$y_p(x) = \int^x \frac{G(t,x)\,b(t)}{a_2(t)} \qquad \overset{\text{Green's Function}}{G(t,x)} = \frac{y_1(t)y_2(x)-y_1(x)y_2(t)}{y_1(t)\,y_2'(t)-y_1'(t)\,y_2(t)}$$

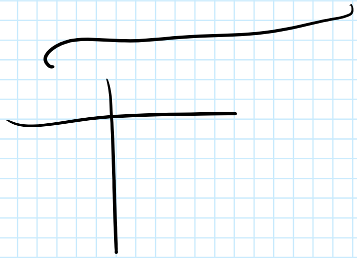
Laplace Transforms ($\mathcal{L}$)

1 Definition:

A laplace transform is a special integral transform:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\,e^{-st}\,dt$$

$$1 \xrightarrow{\mathcal{L}} \frac{1}{s}$$



$$\left.\begin{array}{r} {}_{t \to 0^+} f(t) \\ {}_{t \to \infty} f(t) \end{array}\right\} \text{End Point Theorem}$$

$-a$

⑥ Impulse and Step Function

Step function: Convert piecewise functions into single expression

$$u(t-a) \begin{cases} t<0 \to 0 \\ t \geq a \to 1 \end{cases}$$

Impulse Function

$$I(t-a) \begin{cases} t=a \to \infty \\ t \neq a \to 0 \end{cases}$$

$$\mathcal{L}\{I(t-a)f(t)\} = e^{-as}f(a)$$
$$\mathcal{L}^{-1}\{e^{-as}\} = I(t-a)$$

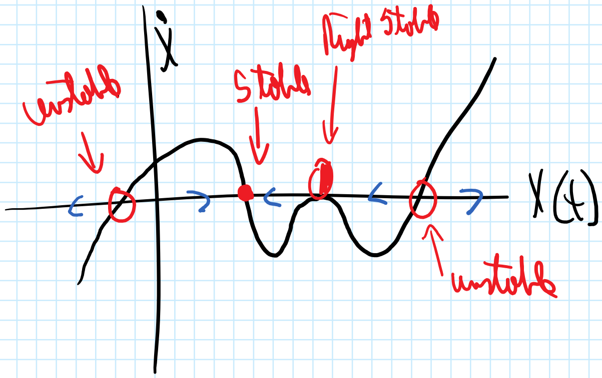
Ⓗ Other Properties

$$\mathcal{L}\{f(at)\} = \frac{1}{a}\mathcal{L}\{f(t)\}_{s \to \frac{s}{a}}$$

Misc. Stuff

① Phase Plot

- Find long term behaviour of function w/o solving ODE given starting point starting point



1. Draw arrows
2. Determine type of ss $\begin{cases} \text{unstable} \\ \text{half stable} \\ \text{stable} \end{cases}$

② Solve Initial Value Problems / Multivalue

$$\left.\begin{aligned} y(x) &= c_1 y_1(x) + c_2 y_2(x) \\ y' &= c_1 y_1'(x) + c_2 y_2'(x) \end{aligned}\right\} \text{given } y(0)=a,\ y'(0)=b$$

Solve for c_1 and c_2 s.t. $y(x)$ fits conditions

(2 equations 2 unknowns)

Multivalue problems are the same, set up a system of equations to solve for constants

③