Review Sheet

Wednesday, April 29, 2020 8:21 PM

Taylors Method

$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ } Power expansion

x_0 is an ordinary point for ODE $\longrightarrow$ $y'' + P(x)y' + Q(x)y = 0$

a_n's are constants

x_0 is an ordinary point of ODE if $\lim_{x \to x_0} P(x)$ and $\lim_{x \to x_0} Q(x)$ exist

What to do?

1. Replace y'', y', y w/ $\sum$ form
2. Try to write everything in terms of $(x-x_0)^n$
   $\rightarrow$ multiply x's into sum to get derivatives into x^n form
3. Change summation limits so first terms are 0
4. Shift series down back to 0
5. Solve for one constant (a_{n+2}) in terms of the other (a_n)

Legendre's ODE and Polynomials

Legendre's ODE:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

n = parameter that takes on non-negative integers

Can use Taylors Method to find a solution

For all n there will be a solution $P_n(x)$ that is a polynomial of order n known as the Legendre polynomial

n → even → $P_n(x)$ contains only $x^{\text{even powers}}$

n → odd → $P_n(x)$ contains only $x^{\text{odd powers}}$

Can find 2^{nd} solution, $Q_n(x)$ which isn't finite at $x = \pm 1$

↳ use whole equation on Final

$P_n(x)$ will be 0 exactly n different places between $-1 < x < 1$

Can re-write Legendre in this Form:

$$a_2(x)\,y_n'' + a_1(x)\,y_n' + a_0(x)\,y_n + \lambda_n^2\, b(x)\, y = 0 \quad \alpha < x < \beta \qquad \text{for a parameter } \lambda$$

$$S(x) = e^{\int \frac{a_1(x)}{a_2(x)}dx} \qquad \overset{\text{weight function}}{w(x)} = \frac{b(x)}{a_2(x)}\, S(x)$$

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$ } can write a function in terms of its Legendre Expansion

$$a_n = \frac{\int_\alpha^\beta f(x)\cdot P_n(x)\, w(x)\, dx}{\int_\alpha^\beta P_n(x)\cdot P_n(x)\, w(x)\, dx}$$ } introduction to "dot product"

Bessels ODE

$$x^2 y'' + (a + 2bx^r)\,x\,y' + \left(c + dx^{2s} - b(1-a-r)\,x^r + b^2 x^{2r}\right) y = 0$$

if $d > 0$

$$y(x) = x^{\frac{1-a}{2}} \cdot \exp\left(-\frac{b \cdot x^r}{r}\right)\left\{ c_1 J_p\left(\frac{\sqrt{d}}{s}x^s\right) + c_2 \overset{\text{or } J_{-p}}{Y_p}\left(\frac{\sqrt{d}}{s}x^s\right)\right\}$$

if $d < 0$

$$y(x) = x^{\frac{1-a}{2}} \cdot \exp\left(-\frac{bx^r}{r}\right)\left\{ c_1 I_p\left(\frac{\sqrt{-d}}{s}x^s\right) + c_2 K_p\left(\frac{\sqrt{-d}}{6}x^s\right)\right\}$$

where $p = \left|\frac{1}{s}\sqrt{\left(\frac{1-a}{2}\right)^2 - c}\right|$

Note, Bessels functions are just like sin/cos and can be found w/ wolfram

Dot Products

Dot Products

$$f \cdot g = \int_{\alpha}^{\beta} f(x) g(x) w(x)\, dx \qquad w(x) > 0 \text{ for all } \alpha < x < \beta$$

"dot product" determined by $w(x)$, α, β

Space can be determined by ODE

A basis set $\{ \varphi_1(x), \dots \varphi_n(x) \}$ is an orthogonal basis if

$$\varphi_n \cdot \varphi_m = 0 \quad \text{for all } m \neq n$$

Sturm Liouville Theory

an ode of the form: $a_2(x) y''(x) + a_1(x) y'(x) + a_0(x) y(x) + \lambda^2 b(x) = 0$ $\quad \alpha < x < \beta$

can be in "self-adjoint" form:

$$\frac{d}{dx}\left(s(x) y'(x) \right) - q(x) y(x) + \lambda^2 w(x) y(x) = 0$$

where

$$s(x) = e^{\int \frac{a_1(x)}{a_2(x)} dx}$$

$$q(x) = \frac{-a_0(x)}{a_2(x)} s(x)$$

Given boundary conditions

$$c_1 y(\alpha) + c_2 y'(\alpha) = 0$$
$$d_1 y(\alpha) + d_2 y'(\beta) = 0$$

Is a Regular Sturm Liouville Problem if

① $s(x)$, $s'(x)$, $q(x)$, $w(x)$ are all continuous on interval $\alpha \le x \le \beta$

② $\left.\begin{matrix} s(x) > 0 \\ w(x) > 0 \end{matrix}\right\}$ for all $\alpha \le x \le \beta$

② $s(x)>0$, $w(x)>0$ } for all $\alpha<x<\beta$

③ $c_1^2+c_2^2>0$, $d_1^2+d_2^2>0$

④ λ^2 appears only in ODE

Properties

① There exist an infinite # of λs that lead to nonzero solutions $\varphi(x)$ to the ODE and BC's

② These λs can be ordered from smallest to largest and are called the eigen values of the RSLP

$$\lambda_1<\lambda_2<\lambda_3 \cdots \quad \lim_{n\to\infty}\lambda_n=\infty$$

③ If λ_ns work they can be replaced w/ λ_n such that φ_n the eigen function associated w/ eigen value λ_n passes through 0 $n-1$ times in interval $\alpha<x<\beta$

when $\varphi_n\cdot\varphi_m \begin{cases} 0 & n\neq m \\ >0 & n=m \end{cases}$

where $f\cdot g = \int_\alpha^\beta f(x)\,g(x)\,w(x)\,dx$

Eigen functions $\{\varphi_1,\varphi_2,\dots\}$ are a complete set of basis functions s.t. if $f(x)$ is any piecewise continuous function on interval $\alpha<x<\beta$ we can express $f(x)$ as $\sum_{n=1}^{\infty} a_n\varphi_n(x)$ where $a_n=\dfrac{\varphi_n\cdot f(x)}{\varphi_n\cdot\varphi_n}$

Sum converges to $\dfrac{f(x^+)+f(x^-)}{2}$

Partial Differential Equations

BVIVP

PDE + BC's + IC

1. The steady state solution, are the PDE / BC's homogenous?

   Yes → skip to step 5

   No → go to step 2

2. Construct ODE and BC's satisfied by time independent solutions

   $y(x,t) \rightarrow y_e(x)$

3. Solve ODE and BC for $y_e(x)$

4. Define the transient solution

   $u(x,t) = y(x,t) - y_e(x) \longrightarrow y(x,t) = u(x,t) + y_e(x)$

5. Use PDE for $y(x,t)$ and plug in

   - get new BC and IC for $u(x,t)$ as well

   (should be homogenous if they weren't already)

6. Solve $u(x,t)$ using separation of variables

   $u(x,t) = \varphi(x) \cdot \gamma(t)$ } plug into PDE

   - get $\varphi(x)$ on one side, $\gamma(t)$ on the other
   - since x and t are independent, only possible if it equals a constant

Solve for $\varphi(x)$ and adapt IC knowing $\gamma(t) \neq 0$

↳ dot product for $\varphi(x)$ (find $w(x)$)

Solve for $\gamma(t)$

combine to get $u(x,t) = \varphi(x) \cdot \gamma(t)$

7. Apply IC using dot product to $u(x,t)$ to find a_n

8. Construct $y(x,t) = y_e(x) + u(x,t)$