Systems of Linear Equations

Consider a System:

$$a_{11}x_1 + \cdots + a_{1m}x_m = b_1$$
$$\vdots \qquad\qquad \vdots$$
$$a_{n1}x_1 + \cdots + a_{nm}x_m = b_n$$

Coefficient Matrix $[n \times m]$ Augmented Matrix $[n \times (m+1)]$

$$\begin{bmatrix} & & \vdots \\ a_1 & \cdots & a_{nm} \end{bmatrix} \qquad \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1m} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nm} & b_n \end{array}\right]$$

Elementary Row Operations

1. Divide / Multiply by a nonzero scalar
2. Subtract/Add a multiple of one row to another
3. Switch two rows

If an augmented matrices can be row reduced to oneanother, then the systems have the same solution set

RREF (Row reduced Echelon Form)

- if a row has nonzero entries, first nonzero entry must be 1
- if a column contains a leading 1, all other entries in that column are 0
- if a row has a leading 1, each row above it w/ a leading 1 must be further left
- Rows of all 0's must be at bottom

Echelon form must just satisfy 1st and 3rd

1's all called pivots, RREF is unique

Types of Solutions to Linear Systems

- No solution (Inconsistent) Row of 0's → nonzero value
- Exactly 1 solution (each column is a pivot)
- Infinite solutions (one free column/variable)

Rank of Matrix

- # of pivot columns
- A is $n \times m$: $\text{rank}(A) \leq n$, $\text{rank}(A) \leq m$

Matrix Operations:

Matrix Addition (1) and Scalar Multiplication (2)

$A, B \in \mathbb{R}^{n \times m}$, $c \in \mathbb{R}$

(1) $A+B = D$ $\quad D_{ij} = A_{ij} + B_{ij}$

(2) $cA = D$ $\quad D_{ij} = c \cdot A_{ij}$

Matrix / Column Vector Multiplication

$Ax = a$ where a_i is dot product of i^{th} row of A w/ x

$\vec{b}$ is a linear combo of vectors $\vec{v}_1 \to \vec{v}_n$ iff $\vec{b} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$

We can write linear system as $[A|\vec{b}]$ as $A\vec{x} = \vec{b}$

Consistent iff $\vec{b}$ is a linear combo of columns of A

Linear Transformations

A map T from $\mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation iff there exists an matrix A $[m \times n]$ s.t.

$T(\vec{x}) = A\vec{x}$ for all $x \in \mathbb{R}$

A transformation T is linear iff

a) $T(x+x') = T(x) + T(x')$ all $x, x' \in \mathbb{R}^n$ } closed under vector addition

b) $T(kx) = kT(x)$ all $x \in \mathbb{R}^n$, $k \in \mathbb{R}$ } closed under scalar multiplication

If T is linear, $T(\vec{0}) = \vec{0}$

If T is linear, there exists a unique matrix A s.t. $T(\vec{x}) = A\vec{x}$

$A = [T]_e$, when we choose coordinate basis, matrix becomes unique

Linear Transformations In Geometry

Rotations $\mathbb{R}^2 \to \mathbb{R}^2$:

Rotation by θ radians in CCW direction

$T(\vec{x}) = A\vec{x}$ $\quad A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

Orthogonal matrix, orientation preserving

Orthogonal Projections in $\mathbb{R}^2$ onto line L

let $\vec{w}$ lie along L

$$\text{proj}_L \vec{x} = \left(\frac{\vec{x}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w}$$

$$T(\vec{x}) = P\vec{x} : P = \frac{1}{w_1^2+w_2^2}\begin{bmatrix} w_1^2 & w_1w_2 \\ w_1w_2 & w_2^2 \end{bmatrix}$$

Reflection About a line in $\mathbb{R}^2$, L

$$\text{ref}_L(\vec{x}) = 2\,\text{proj}_L(\vec{x}) - \vec{x} = 2(\vec{x}\cdot\vec{u})\vec{u} - \vec{x}$$

Matrix is of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ where $a^2+b^2=1$

Shear Matrix Etc on Page 70 of TB

Matrix Multiplication

$\underset{n\times p}{B}$, $\underset{q\times m}{A}$ $B\cdot A$ exists iff $p=q$

$A\cdot B$ exists iff $n=m$

BA is linear transformation $T(\vec{x}) = B(A\vec{x}) = (BA)\vec{x}$

iff $P = P^2$ and $P \neq 0, I$ P is a projection

Matrix multiplication

$A\cdot B = D$ D_{ij} = ith row of A • jth column of B

Associative: $(AB)C = A(BC)$

Not Generally Commutative $AB \neq BA$

Distributive $A(B+C) = AB + AC$

$(A+B)C = AC + BC$

Invertibility

$T: \mathbb{R}^n \to \mathbb{R}^m$, $A\,[n\times m]$ } Invertible $\Rightarrow$ $n=m$

$f^{-1}(f(x)) = x$, $f(f^{-1}(y)) = y$

$g(f(x)) = x \quad g: y \to x$ then $g = f^{-1}$

If f is invertible then so is f^{-1}

If A is invertible

- $n = m$
- $\text{rank}(A) = n$
- $\text{RREF}(A) = I_n$

$A \cdot A^{-1} = I = A^{-1}A$

For $A\vec{x} = \vec{b}$ if $\begin{cases} A \text{ is invertible} \to \text{a unique solution for all } \vec{b} \\ A \text{ is not invertible} \to \text{either } \infty \text{ or } 0 \text{ solutions} \end{cases}$

$A\vec{x} = \vec{0}$ if $\begin{cases} A \text{ is invertible} \to \text{only trivial solution} \\ \text{Non invertible} \to \text{infinitely many solutions} \end{cases}$

Find A^{-1}? $(A \mid I_n) \xrightarrow{\text{Row Reduce}} (I_n \mid A')$ $A' = A^{-1}$

Can justify by showing row operations are elementary matrices which are invertible

$(E_k \cdot E_{k-1} \cdots \cdot E_1)A = I_n$

$A^{-1} = (E_k \cdot E_{k-1} \cdots \cdot E_1)$

$(AB)^{-1} = B^{-1}A^{-1}$

If $AB = I_n$, $A^{-1} = B$, $B^{-1} = A$, $BA = I_n$

Right Invertible $AD = I_n$ Left Invertible $DA = I_m$

If an inverse to a square matrix exists, it is unique

Subspaces of $\mathbb{R}^n$ and their Dimension

The image of a function is all of the values the function takes in its target space (Range)

$T\vec{x} = A\vec{x} \qquad \text{Im}(A), \text{Im}(T)$

- Properties of Image of Linear Transformation
  - contains $\vec{0}$
  - $\text{Im}(T)$ is closed under vector addition
  - $\text{Im}(T)$ is closed under scalar multiplication

The Kernel of a linear transformation

Solutions to $T(\vec{x}) = A\vec{x} = \vec{0}$ (Portion of range T sends to $\vec{0}$)

$\text{Ker}(T), \text{Ker}(A)$, null space

Properties -

- Zero vector is in Ker

- closed under vector addition
- closed under scalar multiplication

$\ker(A) = 0$ iff

- A $(n \times m)$, $\text{Rank}(A) = m$
  - $m \leq n$
  - $n = m$, A must be invertible

Subspaces of $\mathbb{R}^n$

A subset W of $\mathbb{R}^n$ is a linear subspace of $\mathbb{R}^n$ iff

- $W \ni \vec{0}$
- W closed under vector addition
- W closed under scalar multiplication

Image and Kernel are subspaces

$T : \mathbb{R}^m \to \mathbb{R}^n$

$\ker(T)$ subspace of $\mathbb{R}^m$, $\text{im}(T)$ subspace of $\mathbb{R}^n$

Redundant Vectors

$\vec{v}_1 \to \vec{v}_m$ in $\mathbb{R}^n$

a) v_i is redundant if it can be written as a linear combo of $v_1 \to v_{i-1}$

b) vectors $v_1 \to v_m$ are linearly independent iff there are no redundant vectors $\to$ $\begin{bmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{bmatrix} \vec{x} = \vec{0}$ has only trivial solution

c) Vectors in V, a subspace of $\mathbb{R}^n$ are a basis of V if they are linearly independent and span V

For $\mathbb{R}^n$, $m = n$
if $m < n$ doesn't span
if $m > n$ redundant

The length of any linearly independent vectors in a subspace must be $\leq$ any list of vectors which span the subspace

$$\dim(\ker(AB)) \geq \begin{cases} \dim(\ker(B)) \\ \dim(\ker(A)) \end{cases}$$

$$\text{Rank}(AB) \leq \min\{\text{rank}(A), \text{Rank}(B)\}$$

Rank Nullity Theorem $\quad A$ is $n \times m$

$$m = \dim(\ker(A)) + \dim(\text{im}(A))$$

$m = \text{nullity}(A) + \text{rank}(A)$

Find a basis for column space of A,

1. Take all column vectors and remove redundant vectors
2. Take RREF and columns in original matrix corresponding to leading 1's in RREF(A) are basis vectors

Performing row operations doesn't effect row space

Coordinates

choosing a basis => choosing coordinates

$\vec{v} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$ $c_1 \to c_n$ are coordinates of v for basis $\vec{v}_1 \to \vec{v}_n$

Let V be a subspace of $\mathbb{R}^n$

$v_1 \to v_m$ is a basis of V if you can write everything in V

a a $\underbrace{\text{unique}}_{\text{1. linearly independent}}$ $\underbrace{\text{linear combo}}_{\text{2. span}}$ of $v_1 \to v_m$

2 criteria for a basis

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ is linear and $\mathcal{B}\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis of $\mathbb{R}^n$

$$B = [T]_{\mathcal{B}} = \begin{bmatrix} | & & | \\ [T(v_1)]_{\mathcal{B}} & \cdots & [T(v_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix}$$ } This is unique

$$[T(x)]_{\mathcal{B}} = [T]_{\mathcal{B}} [x]_{\mathcal{B}}$$

Coordinates are linear $\begin{cases} [\vec{x}+\vec{y}]_{\mathcal{B}} = [x]_{\mathcal{B}} + [y]_{\mathcal{B}} \\ [k\vec{x}]_{\mathcal{B}} = k[\vec{x}]_{\mathcal{B}} \end{cases}$

A is standard matrix of T, $T(\vec{x}) = A\vec{x}$, B is $\mathcal{B}$ matrix from above

$AS = SB \qquad B = S^{-1}AS \qquad A = SBS^{-1}$

where $S = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$

Two matrices are similar iff $AS = SB$

- represent same transformation w/ respect to different bases

Rules of Similarity:

- A is similar to itself
- A is similar to B $\Rightarrow$ B is similar to A
- A is similar to B, B is similar to C $\Rightarrow$ A is similar to C

$\mathfrak{B}$ of T is diagonal iff $T(v_j)$ is $\parallel$ to v_j

Linear Spaces

A linear space V is set endowed w/ a rule for addition and scalar multiplication

These operations must satisfy the following 8 rules

$f, g, h \in V \qquad c, k \in \mathbb{R}$

1. $(f+g)+h = f+(g+h)$
2. $f+g = g+f$
3. There is a neutral element n in V s.t. $f+n = f$ for all $f \in V$
   n is unique and denoted by 0
4. For each $f \in V$, there exists a $g \in V$ s.t. $f+g = 0$
   g is unique and denoted by $(-f)$
5. $k(f+g) = kf + kg$
6. $(c+k)f = cf + kf$
7. $c(kf) = (ck)f$
8. $1f = f$

Subspaces

A subset W of a linear subspace V is a subspace iff

a) W ∃ neutral element, 0, of V

b) W is closed under addition

c) W is closed under scalar multiplication

Elements $f_1, \ldots, f_n$ are a basis of V iff all elements if they span V and are linearly independent

$\therefore$ every $f \in V$ can be written as a unique linear combo

$$f = c_1 f_1 + \cdots + c_n f_n$$

$c_1 \to c_n$ are the coordinates of f w/ respect to basis $\mathcal{B} = \{f_1 \ldots f_n\}$

$$L(f) = [f]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \qquad V \to \mathbb{R}^n$$

$\mathcal{B}$-coordinate transformation, $L_{\mathcal{B}}$

Linearity of $L_{\mathcal{B}}$

$$[f+g]_{\mathcal{B}} = [f]_{\mathcal{B}} + [g]_{\mathcal{B}}$$

$$[kf]_{\mathcal{B}} = k[f]_{\mathcal{B}}$$

How to find a basis of a linear space (Summary 4.1.6 pg 174)

Linear differential equations

Solution of $f^{(n)}(x) + a_{n-1} f^{(n-1)}(x) + \cdots + a_1 f'(x) + a_0 f(x) = 0$

form an n-dimensional subspace of C^∞

"n^{th} order constant coefficient linear diff-eq"

Consider two spaces, V, W a function T from V to W is a linear transformation iff $\underbrace{T(f+g) = T(f) + T(g)}_{\text{Sum Rule}}$ and $\underbrace{T(kf) = kT(f)}_{\text{Constant multiple rule}}$

$\underbrace{\mathrm{Im}(T) = \{T(f) : f \in V\}}_{\text{Subspace of } W}$ $\underbrace{\mathrm{Ker}(T) = \{f \in V : T(f) = 0\}}_{\text{Subspace of } V}$

$$\dim(V) = \mathrm{rank}(T) + \mathrm{ker}(T) = \dim(\mathrm{Im}(T)) + \dim(\mathrm{Ker}(T))$$

Isomorphism: An invertible linear transformation T is called an isomorphism

Linear space V is isomorphic to W if there exists an isomorphism T from V to W

Coordinate Transformations are isomorphisms:

$f \in V$, $[f]_B \in \mathbb{R}^n$ $\therefore L_B$ is an isomorphism

Properties of Isomorphisms

T from $V \to W$ is an isomorphism iff $\mathrm{Im}(T) = W$, $\mathrm{Ker}(T) = 0$

If V and W are finite dimensionals

T is an isomorphism iff $\dim(V) = \dim(W)$

Matrix of Linear Transformations

– B matrix of a linear transformation (definition 4.3.1 pg 183)

$$B = \{f_1, \dots f_n\}$$

$$[T(f)]_B = \begin{bmatrix} | & & | \\ [T(f_1)]_B & \cdots & [T(f_n)]_B \\ | & & | \end{bmatrix}$$

Change of Basis Matrix

Change of basis from B to A, $S_{B \to A}$

$$B = \{b_1, \dots, b_n\}$$

$$S_{B \to A} = \begin{bmatrix} | & & \\ [b_1] & \cdots & [b_n] \end{bmatrix}$$

$$S_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} | & & | \\ [b_1]_{\mathcal{A}} & \cdots & [b_n]_{\mathcal{A}} \\ | & & | \end{bmatrix}$$

Change of Basis in a subspace of $\mathbb{R}^n$

$\mathcal{A} = \{a_1, \ldots, a_n\}$ $\mathcal{B} = \{b_1, \ldots, b_n\}$

$$\begin{bmatrix} | & & | \\ b_1 & \cdots & b_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} S_{\mathcal{B}\to A}$$

A is $\mathcal{A}$ matrix, B is $\mathcal{B}$ matrix, S is coordinate transformation $\mathcal{B}\to\mathcal{A}$

$AS = SB$ $A = SBS^{-1}$ $B = S^{-1}AS$

Orthogonality and Least Squares

Two vectors $\vec{v}, \vec{w}$ are $\perp$ if $\vec{v}\cdot\vec{w} = 0$

The length (mag/norm) of a vector $\vec{v}$ in $\mathbb{R}^n$, $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$

A vector $\hat{u}$ in $\mathbb{R}^n$ is called a unit vector if $\|\hat{u}\| = 1$

Vectors $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_m \in \mathbb{R}^n$ are orthonormal if they are all unit vectors and orthogonal to eachother

$$\hat{u}_i \cdot \hat{u}_j \begin{cases} i = j \to 1 \\ i \neq j \to 0 \end{cases}$$

Properties of orthonormal vectors $\vec{u}_1 \to \vec{u}_m$

- linearly independent
- form a basis of $\mathbb{R}^m$

Orthogonal Projection

$x \in \mathbb{R}^n$ and a subspace V of $\mathbb{R}^n$

$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$ where $\vec{x}^{\parallel} \in \vec{V}$ } Unique representation

$\vec{x}^{\perp} \perp V$

$\vec{x}^{\parallel}$ is called the orthogonal projection of $\vec{x}$ onto V

$T(\vec{x}) = \mathrm{proj}_V \vec{x} = \vec{x}^{\parallel}$ is linear

Let $\hat{u}_1 \to \hat{u}_m$ be an orthonormal basis of V (subspace of $\mathbb{R}^n$)

$\mathrm{proj}_V \vec{x} = (\hat{u}_1 \cdot \vec{x})\hat{u}_1 + \dots + (\hat{u}_m \cdot \vec{x})\hat{u}_m$

Consider orthonormal basis $\hat{u}_1, \dots, \hat{u}_n$ in $\mathbb{R}^n$

$\vec{x} \in \mathbb{R}^n$: $\vec{x} = (\hat{u}_1 \cdot \vec{x})\hat{u}_1 + \dots + (\hat{u}_n \cdot \vec{x})\hat{u}_n$

Orthogonal Complement

consider subspace V, orthogonal complement $V^{\perp}$ is the set of vectors $\perp$ to all vectors in V

$V^{\perp}$ is the ker of the orthogonal projection onto V

Properties of $V^{\perp}$

- $V^{\perp}$ is a subspace of $\mathbb{R}^n$
- intersection of V and $V^{\perp}$ is $\vec{0}$
- $\dim(V) + \dim(V^{\perp}) = n$
- $(V^{\perp})^{\perp} = V$

Pythagorean Theorem

$\|x+y\|^2 = \|x\|^2 + \|y\|^2$ holds only if $\vec{x}$ and $\vec{y}$ are $\perp$

Inequality for magnitude of projection

$\|\mathrm{proj}_V \vec{x}\| \leq \|\vec{x}\|$

Cauchy Schwartz Inequality

$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$ (only = if vectors are $\parallel$)

Angle between two vectors

$\cos\theta = \dfrac{\vec{x} \cdot \vec{y}}{}$

$$\cos\theta = \frac{x \cdot y}{\|\vec{x}\|\|\vec{y}\|}$$

Correlation Coefficient

$$r = \cos\theta = \frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|\|\vec{y}\|}$$ for two deviation vectors of two different characteristics

Gram-Schmidt Process

Consider a basis $\vec{v}_1 \to \vec{v}_m$ of a subspace V of $\mathbb{R}^n$

for $j = 2, \dots, m$ we resolve the vector v_j into its components $\parallel$ and $\perp$ to the span of the preceding vectors $\vec{v}_1 \to \vec{v}_{j-1}$

$$\vec{v}_j = \vec{v}_j^{\parallel} + \vec{v}_j^{\perp}$$ w/ respect to $\vec{v}_1 \to \vec{v}_{j-1}$

Then

$$\hat{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1, \quad \hat{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|}\vec{v}_2^{\perp}, \quad \dots, \quad \hat{u}_m = \frac{1}{\|\vec{v}_m^{\perp}\|}\vec{v}_m^{\perp}$$

Where $\vec{v}_j^{\perp} = \vec{v}_j - \vec{v}_j^{\parallel} = \vec{v}_j - (\hat{u}_1 \cdot \vec{v}_j)$

QR Factorization

- represents a change in basis from old basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_m\}$ to a new orthonormal basis $\mathcal{A} = \{\hat{u}_1, \dots, \hat{u}_m\}$

R is transformation from $\mathcal{B}$ to $\mathcal{A}$

$$\underbrace{\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}}_{M} = \underbrace{\begin{bmatrix} | & & | \\ \hat{u}_1 & \cdots & \hat{u}_n \\ | & & | \end{bmatrix}}_{Q} R$$

Consider an $n \times m$ matrix M w/ linearly indp. columns $\vec{v}_1 \to \vec{v}_m$

There exists an $n \times m$ matrix Q whose columns $\hat{u}_1 \to \hat{u}_m$ are orthonormal and an upper triangular matrix R w/ positive diagonal entries s.t.

$$M = QR, \text{ this is unique}$$

Furthermore $r_{11} = \|\vec{v}_1\|$, $r_{jj} = \|\vec{v}_j^{\perp}\|$, $r_{ij} = \vec{u}_i \cdot \vec{v}_j$

Computing QR factorization

$r_{11} = \|\vec{v}_1\|$, $\vec{u}_1 = \frac{1}{r_{11}} \vec{v}_1$;

$r_{12} = \vec{u}_1 \cdot \vec{v}_2$, $\vec{v}_2^{\perp} = \vec{v}_2 - r_{12} \vec{u}_1$, $r_{22} = \|\vec{v}_2\|$, $\vec{u}_2 = \frac{1}{r_{22}} \vec{v}_2^{\perp}$;

$r_{13} = \vec{u}_1 \cdot \vec{v}_3$, $r_{23} = \vec{u}_2 \cdot \vec{v}_3$, $\vec{v}_3^{\perp} = \vec{v}_3 - r_{13} \hat{u}_1 - r_{23} \hat{u}_2$, $r_{33} = \|\vec{v}_3^{\perp}\|$, $\hat{u}_3 = \frac{1}{r_{33}} \vec{v}_3^{\perp}$;
