Math Midterm 3 Review

Wednesday, April 17, 2019 2:00 PM

The ODE Test

8.1 Linear ODE's

A linear differentiable equation is a differentiable equation that can be written as $Ly = f$ for some differentiable operator, L

A linear differentiable operator is a linear map $L: C^k([a,b]) \to C^0([a,b])$

Ex) $Ly = (D^2 - Dx + 5)y = y'' - xy' + 5y$

* Order of an ODE (n) refers to the highest derivative present

8.2 Constant Coefficient / Homogenous Linear ODE's

If $f = 0$ in $Ly = f$, the equation is homogonous

Consider $L = D^n + a_1 D^{n-1} + \dots a_n$ where $a_1 \to a_n$ are all constants

This is a constant coefficient ODE

- called a polynomial differentiable operator $P(D)$

Fact, in general linear differentiable operators don't commute, but polynomial ones do!

Ex) $y'' + 5y' + 6y = 0 \longrightarrow P(D) = D^2 + 5D + 6$

If you can factor $P(D)$, such that $(D - r_i)^{m_i} y = 0$

$e^{r_i x}$ is a solution, if root has multiplicity, add x's in front $xe^{r_i x}, x^2 e^{r_i x} \dots$

* For complex conjugate roots, $a + bi$

$e^{ax}\cos bx$ and $e^{ax}\sin bx$ are solutions

General Solution to ODE's

$y(x) = y_h(x) + y_p(x)$

y_h: general soln to homogonous equation

y_p: any particular solution to $Ly = f$

The set of solutions to the homogonous equation ($Ly = 0$) (i.e. $\mathrm{Ker}(L)$) is a Vector Space of dimension n ← order of ODE

This is the reason why the general solution to the homogonous equation is just linear combinations of the basis vectors... you are constructing a V.S.

- With initial conditions, can solve for constants in general equation to find solution

8.3 Inhomogonous / Annihilator Method

Annihilator Method: (For constant coefficient linear ODE)

Find polynomial operator such that $A(D)\,F = 0$

$\therefore A(D) \cdot P(D) = 0$

Can find y_p by solving for Ker of new operator

1. Ignore homogonous solutions
2. Set left over solutions = to $F(x)$
3. Plug $y_p(x)$ back into ODE
4. Solve for unknown constants that make $y_p(x)$ a particular solution

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Spring Mass system (added at the end)

8.8 Cauchy Euler (Equidimensional)

8.8 Cauchy Euler (Equidimensional)

An ODE w/ the form:

$x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \dots + a_n y = 0$

Power of x is always = to derivative of y

Solution must be in form x^r

Plug into equation and solve for r's

If repeated roots, add factors of $\ln(x)$ to solution

$(\ln(x)\, x^r, \ln(x)^2 x^r, \dots)$

8.9 Reduction of Order

If you know 1 homogenous solution, you can use this method to find all solutions

*Most useful for 2^{nd} order ODE's, reduce to 1^{st} order which we can solve
If we reduce a higher order ODE we don't necessarily know how to solve the reduced order ODE (Ex 4^{th} order $\to$ 3^{rd} order)

Say solution $\to$ $y_1(x)$

$y(x) = \underset{\text{some function}}{u(x)} \cdot y_1(x)$ } — Plug into equation and solve

$y'(x) = \dots$

$y''(x) = \dots$

u terms should all cancel, set $w = u'$

$\therefore$ equation becomes 1^{st} order and we can solve (separation of variables or integrating factor)

Once you have w, solve for u $(w = u')$

Then plug u back into $y(x) = u(x) \cdot y_1(x)$ to get full solution, $y(x)$

8.7 Variation of Parameters

$Ly = F$

Suppose $y_1, y_2, \dots, y_n$ is a basis of solution to the homogenous equation $(Ly = 0)$

There is always a particular solution, y_p such that } Need full homogenous solution

$y_p(x) = u_1(x) y_1 + u_2(x) y_2 + \dots + u_n(x) y_n$

Where you can always find $u_1 \to u_n$ via this system:

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \vdots \\ y_1^{(n-1)} & \cdots & & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ F(x) \end{pmatrix}$$

9.1 First Order Linear Systems

$$x_1'(t) = a_{11}(t)\,x_1(t) + a_{12}(t)\,x_2(t) + \dots + a_{1n}\,x_n(t) + b_1(t)$$
$$\vdots$$
$$x_n'(t) = a_{n1}(t)\,x_1(t) + \dots + a_{nn}(t)\,x_n(t) + b_n(t)$$

Where a_{ij} are continuous

If $b_1(t) \to b_n(t) = 0$ the system is homogenous

Write in vector / matrix form

$$\vec{X} = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \qquad \vec{X}'(t) = \begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}$$

$$\vec{X}'(t) = \underbrace{\begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix}}_{A(t)} \vec{X}(t) + \underbrace{\begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix}}_{\vec{b}(t)}$$

9.2

Ways to solve for homogenous solutions

1. Get eigen vectors

diagonalizable

defective

9.4

Find all eigen vectors all of the solutions are of the form $e^{\lambda t}\vec{v}$

Solutions for a given λ and v

eigen value

eigen vector

$$\vec{y}_1 = C_1 \cdot e^{\lambda t} \cdot \vec{v}$$

9.5

Find J.C. form and cycles, answers are slightly more complex and based off of cycles

$$\left\{ \begin{matrix} (A-\lambda I)^{k-1} v \leftarrow \text{general eigen vector} \\ \text{Cycle} \\ v_0 \dashrightarrow v_n \end{matrix} \right\}$$

$$\vec{y}_1 = C_1 V_0 e^{\lambda t} + C_2 e^{\lambda t}(V_1 + t V_0) + C_3 e^{\lambda t}\left(V_2 + V_1 t + V_0 \frac{t^2}{2}\right) \dots$$

Complex eigen vectors / eigen values

Always come in pairs. $(a \pm bi)$

1. take 1 eigen vector and multiply by Eulers Formula $(\cos bt + i \sin bt)$
2. Seperate real/imaginary components into the two different solutions
3. Multiply by e^{at} and C_1 or C_2

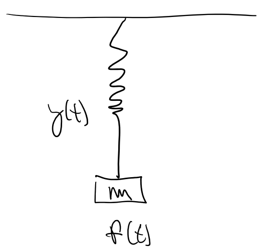
9.8 Matrix Exponential W/ Systems

Fact: e^{tA} gives a basis of solutions to homogenous system of ODE's

- Say you have vector of initial conditions, $\vec{X}(0) = \vec{X}_0$

Solved homogenous system $(\vec{X}(t)) = (e^{tA})\vec{X}_0$

Application of Spring Mass



I No damping: $y''(t) + \omega_0^2 y(t) = 0$

$\omega_0^2 = \frac{k}{m}$

Solutions: $C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$

II With damping $\left(r^2 + \frac{c}{m} r + \frac{k}{m}\right) y = 0$ where $r = \dfrac{-\frac{c}{m} \pm \sqrt{\frac{c^2}{m^2} - 4\frac{k}{m}}}{2}$

Three Cases

1. Underdamping: $c^2 < 4km$ (Negative descriminant)

   - 2 complex conjugate roots

   Solutions: $e^{-\frac{c}{2m}t}(C_1 \cos \mu t + C_2 \sin \mu t)$ where $\mu = \dfrac{\sqrt{4km - c^2}}{2m}$

2. Critical damping: $c^2 = 4km$ (Zero descriminant)

   - 2 repeated real roots

   Solutions: $C_1 e^{-\frac{c}{2m}t} + C_2 t e^{-\frac{c}{2m}t}$

3. Overdamping: $c^2 > 4km$ (positive descriminant)

   - 2 unique real roots

   Solutions: $e^{-\frac{c}{2m}t}(C_1 e^{\mu t} + C_2 e^{-\mu t})$ where $\mu = \sqrt{c^2 - 4km}$

→ 2 unique real roots

Solution: $e^{-\frac{c}{2m}t}\left(C_1 e^{\mu t} + C_2 e^{-\mu t}\right)$ where $\mu = \frac{\sqrt{c^2 - 4km}}{2m}$