Section 4.6: Bases and Dimensions

- Finding a basis for Ker(A) * Note, Ker(A) is always a V.S.
  1. Take r.r.e.f and solve systems
  2. write all equations as combinations of free variables
  3. These combos are the $\vec{v}$ that span the basis

  OR "eye the Kernel"

- Column Space (colspace(A))
  - a span of $(v_1, \dots, v_k)$ where $A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{bmatrix}$
    1. Compute r.r.e.f. (need to find pivot columns)
    2. Go back to original matrix, columns corresponding w/ pivot columns span the colspace(A)

* Row Space
  - conserved w/ row reduction
  - take r.r.e.f.
  - linearly independent rows span row-space (A)

Section 4.8: Rank Nullity Theorem

$\dim(\text{colspace}(A)) + \dim(\text{Ker}(A)) = n$ ← total # of columns

$\text{Rank} + \text{Nullity} = n$

Pivot Columns + Free Variables = n

Section 4.7: Change of Basis

$\mathcal{B}_1 = [v_1, \dots, v_n]$
$\mathcal{B}_2 = [w_1, \dots, w_n]$
both a V.S. for V

How can we relate $[u]_{\mathcal{B}_1}$ w/ $[u]_{\mathcal{B}_2}$

$$[u]_{\mathcal{B}_1} = \boxed{P_{\mathcal{B}_1 \leftarrow \mathcal{B}_2}} [u]_{\mathcal{B}_2}$$

↳ Change of basis matrix from $\mathcal{B}_2 \to \mathcal{B}_1$

$$P_{\mathcal{B}_1 \leftarrow \mathcal{B}_2} = \begin{bmatrix} | & & | \\ [w_1]_{\mathcal{B}_1} & \cdots & [w_n]_{\mathcal{B}_1} \\ | & & | \end{bmatrix}$$

Basis vectors of $\mathcal{B}_2$ (starting basis) in terms of $\mathcal{B}_1$ (ending basis)

Fact: $P_{\mathcal{B}_2 \leftarrow \mathcal{B}_1} = \left[P_{\mathcal{B}_1 \leftarrow \mathcal{B}_2}\right]^{-1}$

inverse to go in reverse order

Section 4.9: Invertible Matrix Theorem

- Key idea, there are a bunch of ways to convey the same idea, that a matrix is invertible

A has n pivots
A has 0 free variables
A has rank n
$Ax = 0$ has only the trivial solution
dimension of the Ker(A) = 0
$\det(A) \neq 0$
$0 \neq \lambda$ of A

$\therefore$ A is invertible

dimension of the $\ker(A) = 0$
$\det(A) \neq 0$
$0 \neq \lambda$ of A
Rowspace / Colspace spans $\mathbb{R}^n$

$\bullet\bullet$ A is invertible

Section 6.1: Linear Transformations

Given V, W (Vector Spaces) a function
$T: V \to W$ is called a linear transformation if
$v_1, v_2 \in V$ $\quad T(v_1+v_2) = Tv_1 + Tv_2$ (closed under addition)
$c \in \mathbb{R}$, $v_1 \in V$ $\quad T(cv_1) = cTv_1$ (closed under scalar multiplication)

Section 6.5

Linear transformations can be written as matrix multiplication
$$T(v) = A \cdot v \quad \text{where } A = \begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{bmatrix}$$
Linearly transform all of the basis vectors

2D Rotation Matrix $= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Section 7.1: Eigenvalue / Eigenvector

Problem: Given linear transformation $T: V \to V$ what is the best basis of V, such that $[T]_B^B$
what is easiest matrix to work with? diagonal matrix?

Eigen Values (λ) / Eigenvectors

Given $T: V \to V$, a non zero vector $\vec{v}$ is an eigen vector if $Tv = \lambda v$

where λ is the eigen value corresponding to $\vec{v}$

If V has a basis of eigen vectors, it is called an eigen basis

Section 7.2

① How to find e-values
$$\det(A - \lambda I) = 0$$
solve for all possible λ's by solving the characteristic polynomial

② Find eigen vectors
solve $\ker(A - \lambda I)$ for λ's
the vectors in the ker are the eigen vectors

* Not all transformations have an eigen base

$$[T]_C^C = P_{C \leftarrow B} [T]_B^B \, P_{B \leftarrow C}$$

Similar matrices $\Rightarrow$ A is similar to B if you can write
$B = S^{-1} A S$

Section 7.3: Diagonalization

What basis should we choose for $[T]_{\beta}^{\beta}$ to be as easy to work with as possible?

$[T]_{\beta}^{\beta}$ is diagonal IFF β is a basis of eigen vectors

A linear map $T: \mathbb{C}^N \to \mathbb{C}^N$ is diagonalizable if there is a basis for $\mathbb{C}^N$ consisting of eigen vectors of T

$$D = S^{-1} \cdot A \cdot S \quad \text{or} \quad A = S \cdot D \cdot S^{-1}$$

D: diagonal matrix of eigen values $\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{bmatrix}$

A: original Transformation Matrix

S, S^{-1}: Change of basis from $\mathcal{E} \to$ eigen basis matrices (opposite order)

If $\mathcal{E}$ is standard basis

$$S = \begin{bmatrix} | & | & & | \\ V_1 & V_2 & \cdots & V_N \\ | & | & & | \end{bmatrix}$$ where V are eigen vectors

$\therefore$ D and A are similar

Facts: A is an $n \times n$ with e-values $\lambda_1 \dots \lambda_n$

$$Tr(A) = \sum_{i=1}^{n} \lambda_i$$

$$\det(A) = \prod_{i=1}^{n} \lambda_i = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$$

The algebraic multiplicity of an eigen value is always $\geq$ the geometric multiplicity

algebraic multiplicity is power of $(\lambda - \lambda_i)$ *

geometric multiplicity is $\dim(\ker(A - \lambda I))$

Section 7.4: Matrix Exponentials

$$e^{At} = S \cdot e^{Dt} \cdot S^{-1} \quad \text{for} \quad A = S \cdot D \cdot S^{-1}$$

$$e^{At} = S \cdot \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} S^{-1}$$

For diagonalizable matrices

For JC form, $e^{Jt} \to$ split into Jordan blocks

Exponential Jordan Block size K $= \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} & \cdots & \frac{t^{K-1}}{K!}e^{\lambda t} \\ & \ddots & & & \vdots \\ & & & \ddots & e^{\lambda t} \end{bmatrix}$

Section 7.6 Jordan Canonical Form

- what if matrix isn't diagonalizable? whats the next simplest form?

Jordan Block

for our class, always an eigen value

Jordan Block

- block of size K corresponding to a # λ is the $K \times K$ matrix $\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$

for our class, always an eigen value

λ's on diagonal
1's on super diagonal (above diagonal)
0's everywhere else

* 1×1 Jordan blocks exist and looks like $[\lambda]$

Jordan Canonical Form

- a matrix is in Jordan Canonical form if A is a block diagonal matrix consisting of only Jordan Blocks.

For EVERY linear map T, there exists a basis β such that $[T]_\beta^\beta$ is in Jordan Form

* The Jordan form is unique up to permutations of the blocks

Generalized eigen vectors

- non-zero vector $\vec{v} \in \mathbb{C}^n$ such that $(A-\lambda I)^K \vec{v} = 0$
- normal eigen vectors are generalized eigen vectors where $K=1$

Cycle length

- cycle length K, for generalized e-vectors generated by $\vec{v}$ is

$$\{(A-\lambda I)^{K-1}\vec{v}, \dots, (A-\lambda I)\vec{v}, \vec{v}\}$$

$$\left.\begin{matrix}(A-\lambda I)^{K-1}\vec{v} \neq 0 \\ (A-\lambda I)^{K}\vec{v} = 0\end{matrix}\right\} \text{Conditions for } \vec{v}$$

Basis (B)

- The basis B is a union of cycles of generalized eigen vectors
- every cycle of length K corresponds to a Jordan block of size K

\# of Jordan Block = # of linearly independent eigen vectors

Section 8.1: Linear Differential Equations

(*) Equation, (IC) set of initial conditions

$$(*) = y^{(n)}(t) + a_1(t)\, y^{(n-1)}(t) + a_2(t)\, y^{(n-2)}(t) + \dots + a_n(t)\, y(t) = F(t)$$

$$(IC) = \{y^{(n-1)}(t_0) = c_1,\ y^{(n-2)}(t_0) = c_2, \dots,\ y(t_0) = c_n\}$$

$c_1 \dots c_n \in \mathbb{R}^n$, $a_1(t) \dots a_n(t)$ and $F(t)$ are all continuous

n = order of (*), if $F=0$ ODE is "homogenous" otherwise inhomogenous

Concept behind solving

Linear Transformation

1. Rephrase (*) as $Ly = f$
2. Solve $Ly = 0$ (find $\ker(L)$)
   - find basis $(y_1, \dots, y_n)$ for $\ker(L)$ so all solutions of $Ly=0$ can be written in terms of the basis $(c_1 y_1 + \dots + c_n y_n)$
3. Show general solution of $Ly = f$ is
   $c_1 y_1 + \dots + c_n y_n + y_p(x)$ any particular solution to $Ly_p = f$
4. Match initial conditions by solving a system (solve for $c_1 \dots c_n$)

- $D: C^1([a,b]) \to C^0([a,b])$
  $\therefore Df = f'$
- $D^k: C^k([a,b]) \to C^0([a,b])$

Note: $C^n([a,b])$ is V.S. of all functions $f: [a,b] \to \mathbb{R}$ with n continuous derivatives

- Linear Differential Operator
  - a linear map $L: C^k([a,b]) \to C^0([a,b])$
    $L = D^k + a_{k-1}(x) D^{k-1} + \dots + a_0$
- Linear differential equation is an equation that can be written in the form $Ly = f$ for some differentiable operator, L
- Existence/Uniqueness

  For an initial value problem, any linear ODE where the coefficients are continuous functions, there exists a unique solution satisfying initial conditions

Section 8.2: Constant Coefficient, Homogeneous L. ODE's

- A linear ODE is a constant coefficient if all of a_i's are constants instead of functions
- Fact, linear differentiable operators in general don't commute but polynomial differentiable operators do

$P(D)y = 0$ want to find basis of $\ker(P(D))$

if ODE is order n, need n linearly independent vectors for the basis

$(D-r)y = 0$

(1) basis $= e^{rx}$ Gen solution $= c_1 e^{rx}$

$(D-r)^m y = 0$

For order two

$b^2 - 4ac$ >0, $=0$, <0

(2) $c_1 e^{rx} + c_2 x e^{rx} + \dots + c_m x^{m-1} e^{rx}$ when you have a multiplicity of root

$a \pm bi$ are roots

(3) $e^{ax} \cos bx$ and $e^{ax} \sin bx$ are solutions