Starting W/ Material After Midterm 1.

Chapter 5

Section 5.3

Orthogonal Transformations and Orthogonal Matrices

A L.T. from $\mathbb{R}^n \to \mathbb{R}^n$ is called orthogonal if it preserves the length of vectors

$$\|T(\vec{x})\| = \|\vec{x}\| \quad \forall x \in \mathbb{R}^n$$

Iff $T(\vec{x}) = A\vec{x}$ is an $\perp$ T., A is an $\perp$ Matrix

Orthogonal Transformations Preserve Orthogonality

Let T be a $\perp$ L.T. from $\mathbb{R}^n \to \mathbb{R}^n$, $\vec{v} \perp \vec{w}$, $\vec{v}, \vec{w} \in \mathbb{R}^n$

$T(\vec{v}) \perp T(\vec{w})$

Orthogonal Transformations and Orthonormal Bases

a) A L.T. from $\mathbb{R}^n \to \mathbb{R}^n$ is $\perp$ iff

$T(\vec{e}_1), T(\vec{e}_2), \dots, T(\vec{e}_n)$ form an orthonormal basis of $\mathbb{R}^n$

b) An $n \times n$ Matrix A is orthogonal iff its columns form an orthonormal basis of $\mathbb{R}^n$

Products / Inverses

a) product of 2 orthogonal matrices is orthogonal

a) product of 2 orthogonal ...

b) The inverse of an orthogonal matrix is orthogonal

Symmetric Matrix: $A^T = A$

Skew-Symmetric Matrix: $A^T = -A$

Dot product: $\vec{v} \cdot \vec{w} = \vec{v}^T \vec{w}$

Property of $\perp$ Matrix

A is orthogonal iff $A^T A = I_n \Rightarrow A^T = A^{-1}$

$\perp$ Matrices

i) A is $\perp$

ii) $L(\vec{x}) = A\vec{x}$ preserves lengths, $\|L(\vec{x})\| = \|\vec{x}\|$, $\forall x \in \mathbb{R}^n$

iii) Columns of A form an orthonormal basis of $\mathbb{R}^n$

iv) $A^T A = I_n$

v) $A^{-1} = A^T$

vi) A preserves dot product $\vec{v} \cdot \vec{w} = (A\vec{v}) \cdot (A\vec{w})$ $\forall \vec{v}, \vec{w} \in \mathbb{R}^n$

Properties of Transpose

a) $(A+B)^T = A^T + B^T$

b) $(kA)^T = kA^T$

c) $(AB)^T = B^T A^T$

d) $\text{rank}(A^T) = \text{rank}(A)$

e) $(A^T)^{-1} = (A^{-1})^T$

Matrix of an $\perp$ Projection

Subspace V of $\mathbb{R}^n$ w/ orthonormal basis $\vec{u}_1, \ldots, \vec{u}_m$

P of the $\perp$ projection onto V is

$$P = Q \cdot Q^T \quad \text{where} \quad Q = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix}$$

P is symmetric, $P^T = P$

Section 5.4

For any Matrix A, $(\mathrm{Im}(A))^{\perp} = \ker(A^T)$

a) If A is $n \times n$, $\ker(A) = \ker(A^T A)$

b) If A is $n \times m$ w/ $\ker(A) = \{\vec{0}\}$, $A^T A$ is invertible

$\perp$ Projections

Consider a vector $\vec{x} \in \mathbb{R}^n$ and a subspace V of $\mathbb{R}^n$

$\mathrm{proj}_V \vec{x}$ is the vector in V closest to $\vec{x}$ in that

$$\|\vec{x} - \mathrm{proj}_V \vec{x}\| < \|\vec{x} - \vec{v}\| \quad \forall \vec{v} \in V$$

Least Squares Solution

Consider a linear system $\underset{n\times m}{A}\vec{x}=\vec{b}$

$\vec{x}^* \in \mathbb{R}^m$ is considered a least-squares solution of the system if

$\|\vec{b}-A\vec{x}^*\| \le \|\vec{b}-A\vec{x}\| \quad \forall\, \vec{x} \in \mathbb{R}^m$

The Normal Equation

The least squares solution to $A\vec{x}=\vec{b}$ are the exact solutions of the (consistent solution)

$$\underbrace{A^T A\vec{x} = A^T\vec{b}}_{\text{normal equation of } A\vec{x}=\vec{b}}$$

If $\ker(A)=\{0\}$ $A\vec{x}=\vec{b}$ has the following unique least squares solution

$$\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$$

Matrix of an $\perp$ projection

V is a subspace of $\mathbb{R}^n$ w/ basis $\vec{v}_1, \dots, \vec{v}_m$

$$A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}$$

Then the matrix of the $\perp$ proj onto V

$$A(A^TA)^{-1}A^T$$

Section 5.5

Inner Products / Inner Product Spaces

An inner product of a linear space V is a rule that assigns a real scalar denoted by $\langle f, g \rangle$ to any pair of f, g of elements of V. The following properties hold for $f, g, h \in V$, $c \in \mathbb{R}$

a) $\langle f, g \rangle = \langle g, f \rangle$ (symmetry)

b) $\langle f+h, g \rangle = \langle f, g \rangle + \langle h, g \rangle$

c) $\langle cf, g \rangle = c\langle f, g \rangle$

d) $\langle f, f \rangle > 0$ for all nonzero f

A linear space endowed w/ an inner product is known as an inner product space

Norm, Orthogonality

The norm or magnitude of an element f of an inner product space is

$$\|f\| = \sqrt{\langle f, f \rangle}$$

Two elements f, g are $\perp$ iff

$$\langle f, g \rangle = 0$$

Orthogonal Projections

$g_1, \ldots, g_m$ is an orthogonal basis of subspace W of inner product space V

$$\text{proj}_W f = \langle g_1, f \rangle g_1 + \cdots + \langle g_m, f \rangle g_m \quad \forall f \in V$$

Orthonormal Basis of T_n

Let T_n be the space of all trigonometric polynomials of order $\leq n$ w/ inner product $\langle f, g \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, g(t)\, dt$

The the functions $\frac{1}{\sqrt{2}}, \sin(t), \cos(t), \ldots, \sin(nt), \cos(nt)$ form an orthonormal basis of T_n

Fourier Coefficient

If f is a piecewise continuous function on the interval $[-\pi, \pi]$ then the best approx f_n in T_n is

$$f_n = \text{proj}_{T_n} f(t)$$

$$= a_0 + b_1 \sin(t) + c_1 \cos(t) + \ldots + b_n \sin(nt) + c_n \cos(nt)$$

where

$$b_k = \langle f(t), \sin(kt) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin(kt)\, dt$$

$$c_k = \langle f(t), \cos(kt) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos(kt)\, dt$$

$$a_0 = \left\langle f(t), \frac{1}{\sqrt{2}} \right\rangle = \frac{1}{\sqrt{2}\pi}\int_{-\pi}^{\pi} f(t)\, dt$$

b_k, c_k, and a_0 are called the Fourier coefficients of f

$$f_n(t) = a_0 + b_1 \sin(t) + c_1 \cos(t) + \cdots + b_n \sin(nt) + c_n \cos(nt)$$

is the n^{th} order Fourier approximation of f

$$\lim_{n \to \infty} \| f - f_n \| = 0$$

$$\lim_{n \to \infty} \| f_n \| = \| f \|$$

$$\underbrace{\| f_n \|^2 = a_0^2 + b_1^2 + c_1^2 + \cdots + b_n^2 + c_n^2}$$

infinite series converges to $\|f\|^2$

Chapter 7

Section 7.1

Diagonalizable Matrices

L.T. $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^n \to \mathbb{R}^n$

A is diagonalizable iff the B matrix of T w/ respect to some basis is diagonal

A is similar to some diagonal matrix B

$S^{-1}AS = B$ for some invertible S

To diagonalize A means to find S and compute $S^{-1}AS$

Eigen Vectors, Eigen Values, Eigen Bases

L.T. $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^n \to \mathbb{R}^n$

Nonzero vector $\vec{v}$ is an eigen vector of A if

$$A\vec{v} = \lambda\vec{v}$$

for a scalar λ, λ is known as the corresponding eigen value

A basis $\vec{v}_1, \ldots, \vec{v}_n$ for A is called an eigen basis

Eigen Bases + Diagonalization

A is diagonalizable iff there exists an eigen basis

$$S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \quad B = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

will diagonalize A, $S^{-1}AS = B$

conversely if S and B diagonalize A, col of S are eigenbasis and diagonal entries of B are corresponding eigen values

The possible real eigen values of an $\perp$ matrix are ± 1

Discrete Traj / Phase Portraits (look at end of sec 7.1)

Section 7.2

Eigen Values and Dets. Char. Eq.

λ is an eigen value of A iff $\det(A - \lambda I) = 0$

char eq. of A

Eigen Values of a triangular matrix are the diagonal entries

Trace is the sum of the Diagonal Entries, $Tr(A)$

Char Eq of 2x2 Matrix

$$\det(A - \lambda I_2) = \lambda^2 - tr(A)\lambda + \det(A) = 0$$

Char Polynomial

A is $n \times n$, $\det(A - \lambda I_n)$ is an n^{th} degree polynomial of the form

$$(-\lambda)^n + tr(A)(-\lambda)^{n-1} + \dots + \det(A)$$

$f_A(\lambda)$ denotes char. polynomial

Algebraic Multiplicity of an eigen Value

eigen value of A, λ_0 has almu K if λ_0 is a root K times

$$\therefore f_A(\lambda) = (\lambda_0 - \lambda)^K g(\lambda)$$
$$g(\lambda_0) \neq 0$$

Number of Eigen Values

$n \times n$ matrix A has at most n real eigen values, even if they are counted w/ their algebraic multiplicity

If n is odd, A has at least 1 real eigen value

$$\det(A) = \prod_{i=1}^{n} \lambda_i$$ where $\lambda_1 \to \lambda_n$ are all eigen values listed w/ almu

$$tr(A) = \sum_{i=1}^{n} \lambda_i$$

Section 7.3

Eigen Spaces

considers λ_j an eigen value of A

$\ker(A-\lambda I)$ is eigen space associated w/ λ

denoted by E_λ

Geometric Multiplicity

Dimension of eigenspace of a given eigen value

$$\dim(E_\lambda) = \text{nullity}(A-\lambda I) = \text{gemu}(\lambda)$$

Eigen Bases and Gemu

a) Eigen Spaces of A will be linearly independent

$s =$ sum of gemu's of λ's $\quad s \leq n$

b) A is diagonalizable iff gemu λ's adds up to n

An nxn matrix w/ n distinct λ's is diagonalizable

Eigen Values of Similar Matrices (A, B)

a) $f_A(\lambda) = f_B(\lambda)$

b) $\text{rank}(A) = \text{rank}(B) \quad \text{nullity}(A) = \text{nullity}(B)$

c) Matrices A and B have the same eigen values w/ the same almu and gemu

d) $\det(A) = \det(B)$, $\text{tr}(A) = \text{tr}(B)$

Almu vs. Gemu

If λ is an eigenvalue of A

$$\text{gemu}(\lambda) \le \text{almu}(\lambda)$$

Strat For Diagonalization

a) Find λ's

b) Find a basis for each E_λ

c) If dim of ↗ add up to n it is diagonalizable

eigenbasis is concatenation of bases from each eigenspaces. recall

$$S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}, \quad B = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Section 7.5

De Moivre's Formula

$$(\cos(\theta) + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Fundamental Theorem of Algebra

any polynomial $p(\lambda)$ w/ complex coefficients can be written as as a product of linear factors

$$p(\lambda) = k(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$$

for some complex #'s $\lambda_1 \to \lambda_n$ and k (don't need to be distinct)

$\therefore$ polynomial $p(\lambda)$ of degree n has exactly n roots

Complex λ's and rotation scaling matrices

A is a real 2x2 mat w/ $\lambda = a \pm bi$ and $\vec{v} + i\vec{w}$ is an eigen vector for $a+bi$

$$S^{-1}AS = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad S = \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix}$$

A complex nxn matrix has n complex λ's when counted w/ almu

$$Tr(A) = \lambda_1 + \cdots + \lambda_n$$
$$det(A) = \lambda_1 \cdot \cdots \cdot \lambda_n$$

Chapter 8

Section 8.1

Spectral Theorem

a matrix A is orthogonally diagonalizable i.e. there exists an S s.t. $S^{-1}AS = S^TAS$ is diagonal iff A is symmetric $(A = A^T)$

Consider a symmetric Matrix A

$\vec{v}_1, \vec{v}_2$ are eigen vectors w/ distinct λ_1, λ_2

then $\vec{v}_1 \cdot \vec{v}_2 = 0$, $\vec{v}_1 \perp \vec{v}_2$

A symmetric nxn matrix A has n real eigen values if they are counted w/ their almu

Orthogonal Diagonalization of Symmetric Matrix A

a) Find λ's of A and basis for each E_λ

b) Use G.S. to find orthonormal basis of E_λ's

c) Concatenate to get orthonormal eigenbasis

$$S = \underbrace{\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}}_{\text{orthogonal}} \qquad S^{-1}AS \text{ will be diagonal}$$

Section 8.2

Quadratic Forms

A function $q(x_1, x_2, \ldots, x_n)$ from $\mathbb{R}^n$ to $\mathbb{R}$ is called a quadratic form if it is a linear combo of functions of the form $x_i x_j$

A quadratic form can be written as

$$q(\vec{x}) = \vec{x} \cdot A\vec{x} = \vec{x}^T A \vec{x}$$

for a unique symmetric $n \times n$ matrix A

Diagonalizing Quadratic Form

$\mathfrak{B}$ is an orthonormal basis of A matrix of q

w/ $\lambda_1 \to \lambda_n$

$$q(\vec{x}) = \lambda_1 c_1^2 + \ldots + \lambda_n c_n^2$$

c_i are coordinates of $\vec{x}$ w/ respect to $\mathfrak{B}^2$

A is positive definite if $q(\vec{x}) > 0$ for all nonzero $\vec{x}$

A is positive semi-definite if $q(\vec{x}) \geq 0$ for all nonzero $\vec{x}$

└── same for negative def / negative semi-def but $\lambda_i \leq 0$

A is indefinite if $q(\vec{x})$ takes pos and neg values

Eigen Values and Definiteness

A is positive definite iff all λ's > 0

A is positive semi-definite iff all λ's ≥ 0

Principal Submatrices and definiteness

A is positive definite iff det of all submatrices > 0

Principal Axes

consider a quadratic of the form $q(\vec{x}) = \vec{x} \cdot A\vec{x}$

eigenspaces of A are called the principal axes of q

└ will be 1D

Ellipses and Hyperbolas

$C \in \mathbb{R}^2 \quad q(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2 = 1$

$\lambda_1, \lambda_2 \rightarrow \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$ of q

if $\lambda_1, \lambda_2 > 0$, C is an ellipse

if one of $\lambda_1, \lambda_2 < 0$, C is a hyperbola

Section 8.3

Singular Values

singular values of $n \times m$ matrix are $\sqrt{\lambda}$'s of $A^T A$

listed w/ above and in decreasing order

Image of Unit Circle

$L\vec{x} = A\vec{x}$ invertible L.T from $\mathbb{R}^2 \to \mathbb{R}^2$ image of unit circle under L is an ellipse w/ semi-major/minor axes σ_1, σ_2

$L(\vec{x}) = A\vec{x}$ is a L.T. from $\mathbb{R}^m \to \mathbb{R}^n$ there exists an orthonormal basis $\vec{v}_1, \ldots \vec{v}_m$ of $\mathbb{R}^m$ s.t.

a) $L(\vec{v}_1) \to L(\vec{v}_m)$ are $\perp$

b) lengths of $L(\vec{v}_1) \to L(\vec{v}_m)$ are $\sigma_1 \to \sigma_m$ of A

To find $\vec{v}_1 \to \vec{v}_m$ find orthonormal eigenbasis of A^TA

w/ corresponding λ's in decreasing order

Singular Value and Rank

A is $n \times m$ w/ rank r

$\sigma_1 \to \sigma_r$ are nonzero

$\sigma_{r+1} \to \sigma_m$ are zero

Singular Value Decomposition (SVD)

any $n \times m$ matrix A can be written as

$$A = U \Sigma V^T$$

U: $\perp$ $n \times n$ matrix

Σ: $n \times m$ w/ first r diagonal entries as nonzero singular values

V: $\perp$ $m \times m$ matrix

$$A = \sigma_1 \underbrace{\vec{u}_1 v_1^T}_{\text{col of } U \text{ and } V} + \cdots + \sigma_r \vec{u}_r v_r^T$$