Final Review (thiccc af)

Sunday, April 28, 2019 8:12 PM

Midterm One Material

Systems of Equations

Solutions (Trichotomy)

1. No solutions → $0 \neq$ Nonzero Value
2. Infinite solutions → At least 1 row of 0's (free variable)
3. One solution → r.r.e.f. no free variables

Conditions For Solutions to homogenous Equations ($A\vec{x} = 0$)

- zero (only trivial solution)
  - A is invertible
- infinite solutions (at least 1 lin. combo)
  - A is non invertible (free variable)

Row Echelon Form

Basic Row Operations

1. Multiply a row by a constant
2. Switch two rows
3. Add/Subtract a multiple of one row to/from another

Tips For Row Reduction

- Get a 1 in the upper left hand corner
- Repeat this step in the next row at the pivot

R.E.F.

Conditions

- the first nonzero entry in each row is a 1
- each leading entry is in a column to the right of the previous row
- rows with zero elements must be at the bottom below nonzero elements

R.R.E.F.

Conditions

- must be in R.E.F. form
- leading entry in each row is only nonzero entry in its column

Rank

- the rank of a matrix is the number of pivot columns or leading ones in R.E.F.

Notation:

$\text{rank}(A) = C$

* For an $n \times n$ matrix A, if $\text{rank}(A) = n$, A is invertible

Pivots

- a pivot column is a column that corresponds with a column of a leading 1 of a matrix in REF form

Standard Matrix Definitions

Transpose

Definition: switch rows and columns

$\boxed{\text{Notation: } A^T}$

Properties

$\det(A^T) = \det(A)$ ← Because transpose doesn't change the diagonals

∴ if A is invertible, so is A^T

$(AB)^T = B^T \cdot A^T \quad (cA)^T = c \cdot A^T \quad (A+B)^T = A^T + B^T$

Diagonal

Definition: the diagonal values of a square matrix

Trace

Definition: Sum of diagonal entries of a matrix

Diagonal Matrix

Definition: Square matrix where all values not on the diagonal are 0

Symetric Matrix

Definition: Matrix where $A = A^T$

Skew Symetric Matrix

- a matrix where $A^T = -A$
- the diagonals must all be 0 because they don't change when you take the transpose

Matrix Multiplication

$A \cdot B = C$ if A is $m \times n$ and B is $n \times p$, C is $m \times p$

$m \times \underbrace{n \cdot n} \times p = m \times p$

In C, c_{ij} is i^{th} row of A dotted with j^{th} column of B

Properties:

Associative: $(AB)C = A(BC)$

Distributive: $A(B+C) = AB + AC$

In General Matrix multiplication is NOT commutative

$A \cdot \vec{0} = \vec{0} \cdot A = \vec{0}$

$IA = A = A \cdot I$

Identity

- $n \times n$ diagonal matrix with only 1's on the diagonal

Notation $\rightarrow I$

Inverses

How to compute?

- General: Augment Matrix with I and row reduce until left side is I. The matrix on the right side is A^{-1}
- Special 2x2 case

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Properties:

$(AB)^{-1} = B^{-1} \cdot A^{-1}$ The inverse is unique

$(A^T)^{-1} = (A^{-1})^T$ $A\vec{x} = \vec{b}$ and A is invertible then $\boxed{\vec{x} = A^{-1}\vec{b}}$

Similar Statements:

A is invertible

$\Updownarrow$

Linear System $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbb{R}^n$

$\Updownarrow$

homogeneous system has only trivial solution

$\Updownarrow$ $\Updownarrow$ $\Updownarrow$

$\text{rank}(A) = n \Leftrightarrow \text{Ker}(A) = 0 \Leftrightarrow \det(A) \neq 0$

Determinants

Relating determinants to row operations

- Switch a row, multiple det by -1
- Multiply by a constant, multiply det by C^{-1}
- Add/Subtract multiple of rows from one another $\longrightarrow$ No change to det

Properties

- can give you volume of parallelepiped
- $\det(AB) = \det(A) \cdot \det(B)$ $\longleftarrow$ Use this in a lot of proofs
- $\det(A^T) = \det(A)$

- For upper/lower triangular matrix, the det is just the product of the diagonals

2 Strategies

1. Row reduce using rules above until you can get an upper/lower triangular matrix
2. Use minor expansion to calculate det

Vector Spaces

Definition: set of something w/ notion of addition/scalar multiplication

- 8 axioms

Subspace Criteria:

1. Closed under scalar multiplication
2. Closed under vector addition
3. Contains the zero vector

Nullspace (Kernel)

Definition: set of all vectors that satisfy $A\vec{x}=0$

Always a Subspace!!

Span/Linear Independence

Span Definition: Something is "in the span of ..." if it is a linear combination of the vectors that make up the span

Linear Independence Definition:

Vectors are linearly independent if

$c_1v_1 + c_2v_2 + \dots c_nv_n = 0$ ONLY if

$c_1 = c_2 = \dots = c_n = 0$

Otherwise, vectors are linearly dependent

(can rearrange to remove redundant vector from set)

Fact: Linear independent if $\mathrm{Ner}(A)=0$

Bases

Definition: A basis for V is a list $(v_1 \dots v_n)$ which satisfies

1. $\mathrm{Span}(v_1, \dots, v_n) = V$
2. $(v_1, \dots, v_n)$ is linearly independent

Examples of Some Standard Bases:

$\mathbb{R}^2 \to \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$

$\mathbb{R}^3 \to \hat{i}, \hat{j}, \hat{k}$

2x2 Matrices

$\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}$

MIDTERM TWO MATERIAL

Bases and Dimensions

Finding a basis for Ner (A)

1. take r.r.e.f and solve system
2. Write all equations as combinations of free variables
3. These combos are the $\vec{v}$ that span the basis of the Ner

Or "eve" the Kernel

Or "eye" the Kernel

Column Space

Definition: a span of $\{V_1, \ldots, V_n\}$ where $A = [V_1 \ldots V_n]$ (Space spanned by columns of Matrix)

1. Compute r.e.f. (need to find pivot columns)
2. Go back to original matrix, columns that correspond to pivots in the r.e.f. span the colspace

Row Space

Definition: Space spanned by rows of Matrix
- conserved through row reduction

1. Compute r.e.f
2. Rows that contain a pivot are $\vec{v}$ that span rowspace

Rank Nullity Theorem

$$\dim(\text{colspace}(A)) + \dim(\text{Ker}(A)) = n$$ ← Number of Columns

Rank + Nullity = # of Columns

Pivots + Free Variables = # of Columns

Change of Basis

$B_1 = [V_1, \ldots, V_n]$

$B_2 = [W_1, \ldots, W_n]$

Relate $[u]_{B_1}$ with $[u]_{B_2}$

$$[u]_{B_1} = \underbrace{P_{B_1 \leftarrow B_2}}_{\text{Change of Basis matrix from } B_2 \text{ to } B_1} [u]_{B_2}$$

$$P_{B_1 \leftarrow B_2} = \underbrace{\begin{bmatrix} [W_1]_{B_1} & \cdots & [W_n]_{B_1} \end{bmatrix}}_{\text{Basis vectors of } B_2 \text{ (starting basis) written in terms of } B_1 \text{ (ending basis)}}$$

Fact:

$$\underbrace{P_{B_2 \leftarrow B_1} = \left(P_{B_1 \leftarrow B_2}\right)^{-1}}$$

Because multiplying by both bases would give you no change ∴ when you multiply the two together they must equal the inverse ∴ they are reciprocals

Invertible Matrix Theorem

Invertible Matrix Theorem

There are many ways to convey the same idea, that a matrix is invertible

Say $A = n \times n$ matrix

$$\left.\begin{array}{l} A \text{ has } n \text{ pivots} \\ A \text{ has } 0 \text{ free variables} \\ A \text{ has rank } n \\ Ax=0 \text{ has only the trivial solution} \\ \det(A) \neq 0 \\ 0 \text{ is not an eigenvalue of } A \\ \dim(\text{Ker}(A)) = 0 \\ \text{Rowspace}(A) \text{ and Colspace}(A) \text{ span } \mathbb{R}^n \end{array}\right\} \therefore A \text{ is invertible}$$

Linear Transformations

Given V, W (vector spaces) a function $T: V \to W$ is considered a linear transformation if

$V_1, V_2 \in V \quad T(V_1 + V_2) = T(V_1) + T(V_2)$ } Closed under addition

$V_1 \in V \quad c \in \mathbb{R} \quad T(cV_1) = cT(V_1)$ } Closed under scalar multiplication

Linear Transformations can be written as matrix multiplications

$$T(v) = Av \quad \text{where } A = \underbrace{\begin{bmatrix} | & & | \\ T(e_1) & \cdots\cdots & T(e_n) \\ | & & | \end{bmatrix}}_{\text{Linearly transform all of the basis vectors}}$$

$[T]_B^C$ Transforms basis vectors of B and write them in terms of C

2D Rotation Matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Eigenvalues / Eigenvectors

Problem: Given linear transformation $T: V \to V$ what is the best basis of V, such that $[T]_B^B$ is simple (What is easiest matrix to work with? A diagonal matrix)

Definition: given $T: V \to V$ is a linear transformation, a non-zero vector $\vec{v}$ is eigen vector if $Tv = Av = \lambda v$ where λ is the corresponding eigen value

Eigen Basis

Definition: if V has a basis of eigen vectors

* Not all transformations have an eigen basis

How to find eigen values:

$\det(A - \lambda I) = 0$

Solve for all possible λ's by solving the characteristic polynomial

How to find eigen vector

Solve $\text{Ker}(A - \lambda I)$ for all λ's found

Vectors in the Ker are the eigen vectors

$[T]_C^C = P_{C \leftarrow B} \; [T]_B^B \; P_{B \leftarrow C}$

Vectors in the Ker are the eigen vectors

$$[T]_C^C = P_{C \leftarrow B} \, [T]_B^B \, P_{B \leftarrow C}$$

Diagonalization

$[T]_B^B$ is diagonal iff B is an eigen basis

A linear map $T: \mathbb{C}^n \to \mathbb{C}^n$ is diagonalizable if there is a basis for $\mathbb{C}^n$ consisting of eigen vectors of T

$$D = S^{-1} A S \qquad A = S D S^{-1}$$

D: Diagonal matrix of eigen values

A: Original transformation matrix

S, S^{-1}: change of basis Matrix from $C \to$ eigen basis and eigen basis $\to C$

If C is the standard basis

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \qquad S = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \text{ where } v \text{ are the eigen vectors}$$

$\therefore$ D and A are similar

Given A is an $n \times n$ matrix with eigen values $\lambda_1 \to \lambda_n$, **Facts:**

$Tr(A) = \sum_{i=1}^{n} \lambda_i$ (the trace of A is equal to the sum of its eigen values)

$\det(A) = \prod_{i=1}^{n} \lambda_i$ (the det of A is equal to the product of its eigen values)

The algebraic multiplicity of an eigen vector is always $\geq$ geometric multiplicity *

algebraic multiplicity is power of $(C - \lambda_i)$

geometric multiplicity is $\dim(\ker(A - \lambda I))$

Matrix Exponential

$$e^{At} = S \cdot e^{Dt} \cdot S^{-1} \quad \text{given} \quad A = S \cdot D \cdot S^{-1}$$

$$e^{At} = S \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} S^{-1}$$

For diagonalizable matrices

For JC form, split up into Jordan blocks

Exponential Jordan Block for cycle length K = $\begin{bmatrix} e^{\lambda t} & t e^{\lambda t} & \frac{t^2 e^{\lambda t}}{2!} & \cdots & \frac{t^{K-1}}{(K-1)!} e^{\lambda t} \\ & \ddots & & & \vdots \\ & & & & e^{\lambda t} \end{bmatrix}$

Jordan Canonical Form

What if matrix isn't diagonalizable? What is the next best form?

Jordan Block

- block of size K corresponding to a λ is the $K \times K$ matrix with λ's on the diagonal, 1's on the super diagonal, and 0's everywhere else

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & 0 \\ \vdots & & & & 1 \\ 0 & \cdots & & 0 & \lambda \end{pmatrix}$$

* A 1×1 Jordan block exists and is just $[\lambda]$

Jordan Canonical Form

Definition: A matrix (J) is in Jordan Canonical form if J is a block diagonal matrix consisting of only Jordan Blocks

Fact: For every linear map T, there exists a basis B such that $[T]_B^B$ is in Jordan Form. The Jordan form is unique up to permutations of the Jordan Blocks

Generalized Eigen Vector
Definition: nonzero vector $\vec{v} \in \mathbb{C}^n$ such that $(A-\lambda I)^K \vec{v} = 0$
(normal eigen vectors are generalized eigen vectors of cycle length (K) 1)

Cycle Length
Definition: a cycle length K, for a generalized eigenvector $\vec{v}$ is

$$\left\{(A-\lambda I)^{K-1}\vec{v}, \ldots, (A-\lambda I)\vec{v}, \vec{v}\right\}$$

Where $(A-\lambda I)^{K-1}\vec{v} \neq 0$
$(A-\lambda I)^K \vec{v} = 0$

Basis (B)
Definition: The basis B is a union of cycles of generalized eigen vectors
Fact: Every cycle length K corresponds to a Jordan Block size K

MIDTERM 3 MATERIAL/ODES

Linear ODEs

Linear Differential Equation
Definition: a differential equation that can be written as $Ly = f$ for some differentiable operator L

Linear Differentiable Operator
Definition: A linear map $L: C^K([a,b]) \to C^0([a,b])$
Ex $Ly = (D^2 - Dx + 5)y = y'' - y'x + 5y$

Note: The Order (n) of an ODE refers to the highest derivative present in the ODE

Constant Coefficient/Homogenous ODE's

Homogenous Equation
Definition: if $f = 0$ in $Ly = f$ then the equation is homogenous

Constant Coefficient ODE
Definition: Consider $L = D^n + a_1 D^{n-1} + \ldots + a_n$, the ODE is considered constant coefficient if $a_1 \to a_n$ are all constants

$\underbrace{L = D^n + a_1 D^{n-1} + \ldots + a_n}$
P(D), called a polynomial differentiable operator

* In general, Linear differentiable operators don't commute, but polynomial differentiable operators do

How to find solutions to ODE?
Factor P(D) such that $(D - r_i)y = 0$
$e^{r_i x}$ is a solution (if root is repeated, add multiples of x)
$xe^{r_i x}, x^2 e^{r_i x}, x^3 e^{r_i x} \ldots$
For complex conjugate roots, $a \pm bi$
$e^{ax}\cos bx$ and $e^{ax}\sin bx$ are roots

same idea applies for repeated roots

General Solutions to ODE's

$y(x) = y_h(x) + y_p(x)$

y_h: general solution to homogenous equation
y_p: any particular solution to $Ly = f$

Note, the set of solutions to the homogenous equation $(Ly = 0)$ is a vector space of dimension n

n: order of ODE

Reason why the general solution to homogenous equation is just a linear combo of the basis solutions

With initial conditions, you can solve for constants in general equation

(constructing a V.S.)

Inhomogeneous ODE's / Annihilator Method

Annihilator Method (Only for constant coefficient ODE's)

1. Find Polynomial operator such that $A(D)F = 0$

   $\therefore A(D)\cdot P(D) = 0$

2. Find y_p by solving for Ker of new operator

   a. Ignore homogeneous solutions
   b. Set left over solutions $= y_p(x)$ and find derivatives
   c. Plug back into ODE
   d. Solve for unknown constants to fully get $y_p(x)$

Cauchy Euler (Equidimensional ODE's)

Definition: An ODE in the form

$$\underbrace{x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \dots + a_n y = f}_{\text{Power of } x \text{ always = to order of derivative of } y}$$

Homogeneous Solution:

- Must be in form x^r
- plug into equation and solve polynomial that forms in terms of r
- roots are r values for solutions

  $x^{r_1}, x^{r_2}, \dots$ etc

- for repeated roots add factors of $\ln(x)$ $\left(\ln(x)x^r, (\ln(x))^2 x^r, \dots\right)$
- complex roots $a \pm bi$

  $x^a \cos(b\ln(x)) + x^a \sin(b\ln(x))$

Reduction of Order

Method for solving ODE

Use: When you know 1 homogeneous solution, can find entire solution

Most useful for 2nd order ODE's, reduce to 1st order which we can solve

If we reduce a higher order ODE, we won't necessarily know how to solve it

(4th order → 3rd order) doesn't help much

Say one homogeneous solution is $y_1(x)$

$y(x) = u(x)\cdot y_1(x)$ some function

$y'(x) = u'(x)y_1(x) + u(x)y_1'(x)$

$y''(x) = u''(x) + 2u'(x)y_1'(x) + u(x)y_1''(x)$

→ Plug these into ODE and solve

All of $u(x)$ terms should cancel and by setting $w = u'$ you can create a 1st order equation and solve } (Separation of variables / integrating factor)

Then solve for u (remember $u' = w$)

Finally, multiply $u(x)$ by $y_1(x)$ to get full solution

Variation of Parameter (Not Required)

Variation of Parameters (Not Required)

For the ODE $Ly = f$ ← Need full homogenous solution

Suppose $y_1, y_2, \dots, y_n$ is a basis for the homogenous equation ($Ly = 0$)

There is always a particular solution, y_p such that $y_p(x) = u_1(x)y_1 + u_2(x)y_2 + \dots + u_n(x)y_n$

where you can solve for $u_1 \to u_n$ using this system:

$$\begin{pmatrix} y_1 & y_2 & & y_n \\ y_1' & y_2' & & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & \cdots & & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f(x) \end{pmatrix}$$

First Order Linear Systems

$$\text{System} \begin{cases} x_1'(t) = a_{11}(t)x_1(t) + a_{12}(t)x_2 + \cdots + a_{1n}x_n(t) + b_1(t) \\ \vdots \\ x_n'(t) = a_{n1}(t)x_1(t) + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n(t) + b_n(t) \end{cases}$$

* a_{ij} are continuous
* if $b_1 \to b_n$

Write in Vector form:

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \quad \vec{x}'(t) = \begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} \quad A = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix} \quad \vec{b}(t) = \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix}$$

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{b}(t)$$

Ways to solve homogenous solutions:

Get eigen vectors

diagonalizable	deffective

Find all eigen vectors, all of the solutions are of the form $Ce^{\lambda t}\vec{v}$

$$\vec{y}_1 = C_1 e^{\lambda t}\vec{v}$$

Find J.C form and cycles

answers are slightly more complex due to cycles

k length cycle

$$\left\{ (A - \lambda I)^{k-1} v_n \cdots v_n \right\}$$

$$v_0 \longrightarrow v_n$$

$$\vec{y}_1 = C_1 v_0 e^{\lambda t} + C_2 e^{\lambda t}(v_1 + v_0(t)) + ($$

Complex eigen values / vectors (always come in pairs)

take 1 eigen vector and multiply

$$\cdot_3 e^{\lambda t}\left(v_2 + v_1(t) + v_0\left(\frac{t^2}{2!}\right)\right)$$

1. take 1 eigen vector and multiply by Eulers Formula
2. seperate real/imaginary components into the two different solutions
3. Multiply by e^{at} and C_1 or C_2

$\vec{y}_1 = C_1 v_0 e^{\lambda t} + C_2 e^{\lambda t} \left(v_1 + v_0(t) \right) + ($

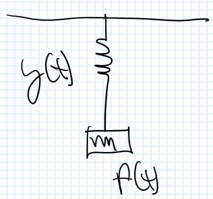
Matrix Exponential W/ systems

Fact : e^{tA} gives a basis of solutions to homogenous systems of ODE's

Say we have a vector of initial conditions, $\vec{X}(0) = \vec{X}_0$

Solved homogenous system $\vec{X}(t) = e^{tA} \cdot \vec{X}_0$

Application of Spring Mass (Slightly out of order)



1. No damping : $y''(t) + \omega_0^2 y(t) = f(t)$ (C_1 damping constant $= 0$)

$\omega_0^2 = \frac{K}{m}$

Homogenous Solutions: $C_1 \cos \omega_0 t + C \sin \omega_0 t$

2. With damping $\left(r^2 + \frac{c}{m} r + \frac{K}{m} \right) y = f(t)$

Where homogenous roots : $r = \dfrac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4\frac{K}{m}}}{2}$

Subcases

1. Underdamping : $c^2 < 4Km$ (Negative descriminant)
   - 2 complex conjugate roots
   - Homogenous Solutions: $e^{-\frac{c}{2m}t} \left(C_1 \cos \mu t + C_2 \sin \mu t \right)$ where $\mu = \dfrac{\sqrt{4Km - c^2}}{2m}$
2. Critical damping : $c^2 = 4Km$ (Descriminant $= 0$)
   - 2 repeated real roots
   - Homogenous Solutions: $C_1 e^{-\frac{c}{2m}t} + C_2 t e^{-\frac{c}{2m}t}$
3. Overdamping: $c^2 > 4Km$ (positive descriminant)

$$\ldots_3 e^{\lambda t}\left(v_2 + v_1(t) + v_0\left(\frac{t^2}{2!}\right)\right)$$

- 2 unique real roots

Homogeneous Solution: $e^{-\frac{c}{2m}t}\left(C_1 e^{\mu t} + C_2 e^{-\mu t}\right)$ $\quad \mu = \sqrt{c^2 - 4km}$

Post Midterm 3 Material

Phase Plane Analysis

$x'(t) = F(x,y)$
$y'(t) = G(x,y)$ $\quad$ Velocity $\quad \vec{X}'(t) = A\,\vec{X}(t) \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Plot solutions, curves in Plane
- Can never intersect due to existence and uniqueness theorem

Equilibrium Points
$\frac{dx}{dt} = \frac{dy}{dt} = 0 = F(x,y) = G(x,y)$

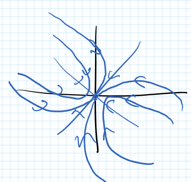
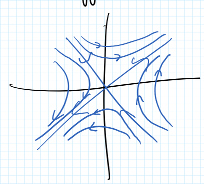
For $\vec{X}' = A\vec{X}$ $\quad (x_0, y_0)$ is an equilibrium point if $\vec{X}(t) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is a constant solution
if $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \operatorname{Ker}(A)$

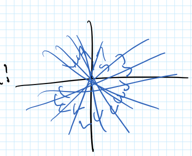
What happens if you wiggle away from eq. Point

1) Stable
eq. point is stable if $\lim_{t\to\infty}$ for all solutions is the eq. point

2) Unstable
eq. point is unstable if $\lim_{t\to\infty}$ not all solutions go towards eq. point

Cases (find λ's of A)

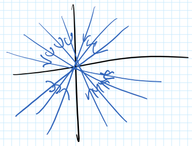
1) Distinct Real eigenvalues
 a) r_1 and $r_2 < 0$
 equilibrium point: node, stable $\quad$ graph: 
 b) r_1 and $r_2 > 0$
 equilibrium point: node, unstable $\quad$ graph: same but w/ arrows in opposite directions
 c) $r_1 < 0 < r_2$
 equilibrium point: saddle, unstable
 graph: 

2) Real Repeated Eigen Values
 a) A is diagonalizable
 I) $\lambda > 0$
 equilibrium point: proper node (unstable) $\quad$ graph: 
 II) $\lambda < 0$

equilibrium point: proper node (unstable)   graph:

(ii) $\lambda < 0$

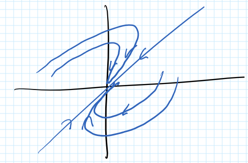
equilibrium point: proper node (stable)   graph



(b) A is defective

(i) $\lambda < 0$

equilibrium point: degenerate node (stable)   graph:



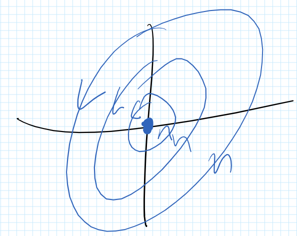
(ii) $\lambda > 0$

equilibrium point: degenerate node: (unstable)   graph: ↑ opposite arrow direction

(3) Complex conjugate eigen values $(\alpha + bi)$

(a) $\alpha = 0$

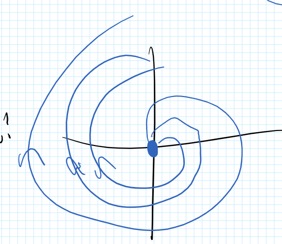
equilibrium point: center (not stable according to lin definition)   graph:



(b) $\alpha > 0$

equilibrium point: spiral point (unstable)

graph:



(c) $\alpha < 0$

equilibrium point: spiral point (stable)   graph: ↑ same but opposite arrows

Phase Portraits of Nonlinear Systems

$\frac{dx}{dt} = F(x,y)$ $\frac{dy}{dt} = G(x,y)$ where $F(x,y)$ and $G(x,y)$ are nonlinear

(x_0, y_0) is an equilibrium point if $\left.\begin{array}{l} \frac{dx}{dt} = F(x_0, y_0) = 0 \\ \frac{dy}{dt} = G(x_0, y_0) = 0 \end{array}\right\}$ at rest

For a nonlinear system, can't really see what overall phase portrait looks like
Near ~~to~~ points (x_0, y_0) you can approximate as linear function (behave similarly)

Take Jacobian: $\begin{pmatrix} \frac{\partial F}{\partial x}(x_0, y_0) & \frac{\partial F}{\partial y}(x_0, y_0) \\ \frac{\partial G}{\partial x}(x_0, y_0) & \frac{\partial G}{\partial y}(x_0, y_0) \end{pmatrix}$

Take Jacobian:
(@ ≈ point)

$$\begin{pmatrix} \frac{\partial F}{\partial x}(x_0,y_0) & \frac{\partial F}{\partial y}(x_0,y_0) \\ \frac{\partial G}{\partial x}(x_0,y_0) & \frac{\partial G}{\partial y}(x_0,y_0) \end{pmatrix}$$

$$\vec{X}'(t) = J(x_0,y_0)\,X(t)$$

Find λ's of Jacobian: Near ≈ point, non linear system acts like linear case w/ same λ characterization