Lecture 1

Wednesday, January 20, 2021 11:47 AM

Masses and Springs VS Rigid Body

Time scales are intrinsic to the dynamic behavior
- finite time it takes for spring to transfer force

Common features

Inertia
Restoring Force
Damping
External Force

Inertia
- a mass / distribution of mass undergoing movement

Restoring Force
- mass is displaced some force acts to restore system towards original config

Damping
- System loses kinetic energy

External Forcing

Degrees of freedom?

- # of coordinates required to define the config of a system at any given instant

Start w/ simple spring mass system

Lecture 2

Monday, January 25, 2021 2:59 PM

Rigid Bodies

- a force in one location is assumed to affect the entire body at once

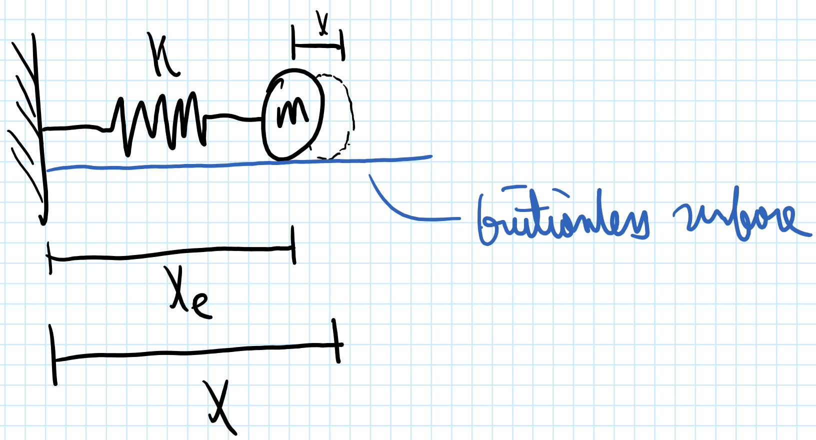
Common Features

Mass Spring System

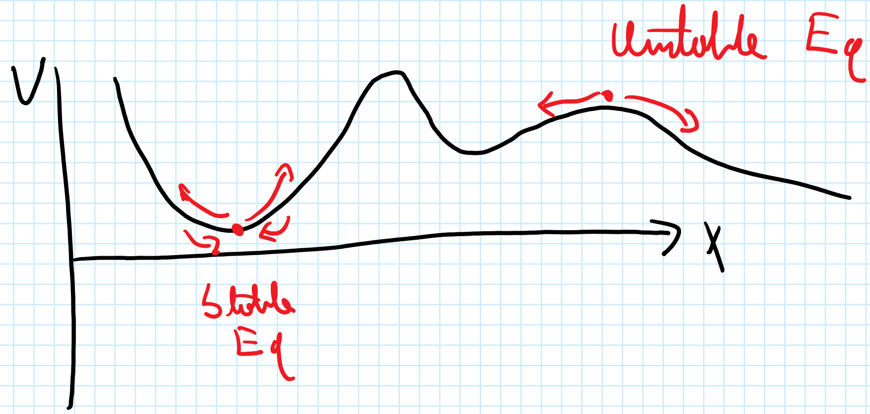
- periodic
- fixed amplitude (w/ some decay)

change in speed (KE isn't constant)

Simple Harmonic Motion (SHM): Model



What do we need to achieve SHM?
How quickly will system oscillate?
What is max speed?
Where will mass be at time t?



Conditions for SHM

1. Need to be at stable equilibrium

$$\frac{d^2V}{dx^2} > 0 \quad \text{(positive curvature)}$$

only needs to be true locally

2. "Simple Harmonic" ⇒ motion follows a single sin/cos w/ single frequency

3. Need a linear restoring force

i.e. Hooke's Law $F = -Kx = -K(x - x_e)$

Hooke's Law

$$F(x) = -Kx \qquad \sum F = ma = m\ddot{x}$$

$$-Kx = m\ddot{x}$$

$$\ddot{x} + \frac{K}{m}x = 0$$

$$\omega_n^2 = \frac{K}{m} \qquad \ddot{x} + \omega_n^2 x = 0$$

ω_n is the angular frequency (units $\frac{\text{rad}}{s}$) at which the mass spring system naturally oscilates

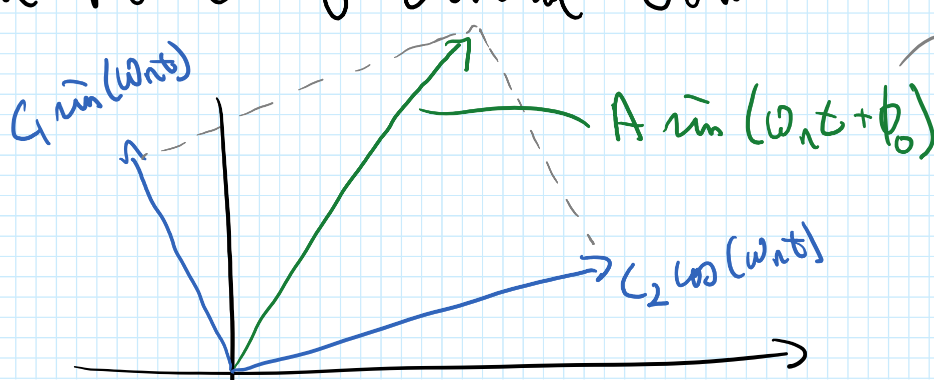
$$x_1 = \sin(\omega_n t), \quad x_2 = \cos(\omega_n t)$$

Linear ODE $\Rightarrow$ linear combos of solutions are solutions

$$x = C_1 \sin(\omega_n t) + C_2 \cos(\omega_n t)$$

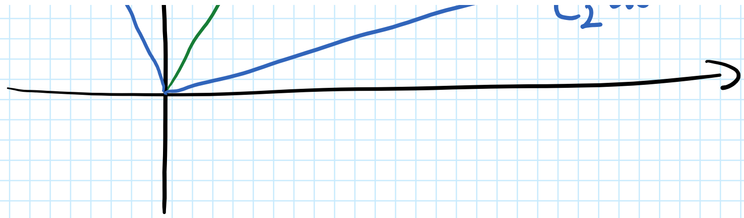
If $x(0) = x_0$, $\dot{x}(0) = 0 \Rightarrow x = X_0 \cos(\omega_n t)$

Useful Form of General Solution



initial phase

$$A = \sqrt{C_1^2 + C_2^2}$$



Period / Frequency

Quantity T is the period (units of time)

$$x(t) = x(t+T) = x(t+nT) \quad n \in \mathbb{Z}$$

$$A\sin(\omega_n t + \phi_0) = A\sin(\omega_n(t+nT) + \phi_0)$$

$$\omega_n t + \phi_0 = \omega_n(t+nT) + \phi_0$$

$$\omega_n t = \omega_n(t+nT)$$

$$\omega_n t + n\omega_n T + \phi_0 - \omega_n t - \phi_0 = 2\pi n$$

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{k}}$$

$$f = \frac{1}{T} = \frac{\omega_n}{2\pi}$$

Notice, T doesn't depend on amplitude

Vertical Mass Spring System

- equilibrium point will change
- no effect on dynamic properties of systems

$$F_y = -F_s$$

$$mg = k(x'_e - x_e)$$

$$\frac{mg}{k} + x_e = x'_e$$

$$x = X - x'_e = X - x_e - \frac{mg}{k}$$

Total Force Acting on the mass:

$$F = -k\left(x + \frac{mg}{k}\right) + mg$$

$$F = -kx$$

Energy and Simple Harmonic Motion

$$F(x) = -\frac{dV}{dx}$$

$$F(x) = -kx \implies V = -\int F(x)\,dx = \frac{1}{2}kx^2$$

The Spring Potential V is conservative (i.e. path independent)

- only depends on the value of x

If the only force applied is conservative, then the entire system is conservative

$$E = PE + KE = \text{constant}$$

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} K x^2$$

If $\dot{x} = 0$, x becomes a maximum

$$x_{max} = A = \sqrt{\frac{2}{K} E}$$

$$\dot{x}_{max} = \sqrt{\frac{2E}{m}}$$

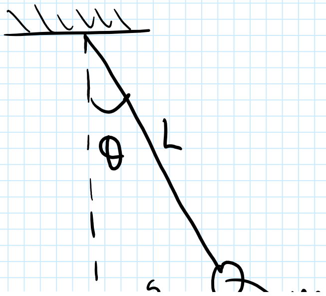
From our general solution

$$\dot{x}_{max} = A \omega_n$$

$$\omega_n = \sqrt{\frac{K}{m}}$$

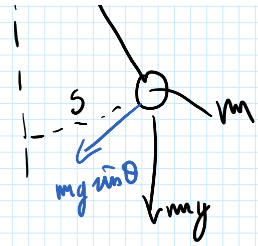
The Simple Pendulum

↳ point mass and massless string



$$m l \ddot{\theta} = -m g \sin(\theta)$$

$$\ddot{\theta} = \frac{-g}{l} \sin(\theta)$$



$$\ddot{\theta} = -\frac{g}{\ell}\sin(\theta)$$

$$s = \ell\theta$$

Linearize θ, $\sin\theta \approx \theta$ (from Taylor Series)

$$\ddot{\theta} \approx -\frac{g}{\ell}\theta$$ } - linearized equation that's good for smaller θ's

Note this is same form as spring mass

$$\omega_n = \sqrt{\frac{g}{\ell}} \qquad f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}, \quad T = 2\pi\sqrt{\frac{\ell}{g}}$$

Period is independent of m and θ_0

Simple Pendulum + Energy

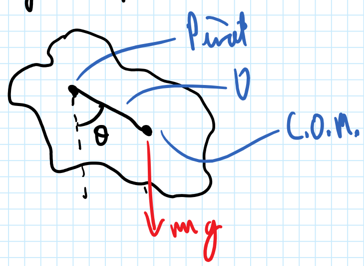
$$V = mgh$$

$$V(\theta) = mg\ell(1-\cos\theta)$$

for small θ, $\cos\theta \approx 1 - \frac{\theta^2}{2}$

$$E_{total} = \frac{1}{2}m\dot{\theta}^2 + \frac{1}{2}\frac{mg}{\ell}\theta^2$$

The Physical Pendulum



Torque around pivot

$$\tau = I\ddot{\theta} = -MgD\sin\theta$$

$$\boxed{\ddot{\theta} = \frac{-MgD}{I}\sin\theta}$$

Again, apply small angle formula

$$\ddot{\theta} = \frac{-MgD}{I}\theta$$

w/ Sinusoidal Solution

$$\omega_n = \sqrt{\frac{M_g D}{I}}$$

Can write ODE in another form:

$$\ddot{x} = -\omega_n^2 x$$

$$x = Ae^{i(\omega_n t + \phi_0)}$$

$$\dot{x} = i A \omega_n e^{i(\omega_n t + \phi_0)}$$

$$\ddot{x} = -A \omega_n^2 e^{i(\omega_n t + \phi_0)}$$

also satisfies ODE shown above

Want real part?

$$Ae^{i(\omega_n t + \phi_0)} = A\cos(\omega_n t + \phi_0) + iA\sin(\omega_n t + \phi_0)$$

$$\mathrm{Re}[x] = \mathrm{Re}\left[Ae^{i(\omega_n t + \phi_0)}\right] = A\cos(\omega_n t + \phi_0)$$

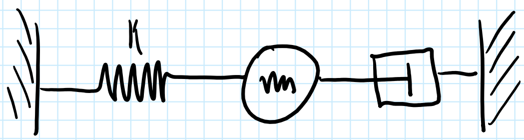
Lecture 4

Monday, February 1, 2021 2:58 PM

Damped Oscillations

- what is cause of damping in our experiment

Linear Viscous Damping



Mass experiences drag, always acts against the direction of motion (velocity)

Proportional to velocity

velocity

$$F_{damping} = -c\dot{x}$$

linearly proportional

$$\sum F = -c\dot{x} - kx = m\ddot{x}$$

$$x = Ae^{\gamma t}$$

$$\dot{x} = A\gamma e^{\gamma t}$$

$$\ddot{x} = A\gamma^2 e^{\gamma t}$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

$$A\gamma^2 e^{\gamma t} + \frac{c}{m}A\gamma e^{\gamma t} + \frac{k}{m}Ae^{\gamma t} = 0$$

$$Ae^{\gamma t}\underbrace{\left(\gamma^2 + \frac{c}{m}\gamma + \frac{k}{m}\right)} = 0$$

$$Xe^{\gamma t}\underbrace{\left(\gamma^2+\frac{c}{m}\gamma+\frac{k}{m}\right)}_{=0}=0$$

$$\gamma^2+\frac{c}{m}\gamma+\frac{k}{m}\gamma=0$$

Roots of Quadratic

$$\gamma=\frac{-\frac{c}{m}\pm\sqrt{\left(\frac{c}{m}\right)^2-4\,\frac{k}{m}}}{2}$$

could be $\begin{cases}\text{real}\\ 0\\ \text{imaginary}\end{cases}$

Define: Critical Damping Value, c_c

If $c=c_c$ then $\frac{\text{discriminant}}{2}\to 0$

$$\left(\frac{c_c}{2m}\right)^2-\frac{k}{m}=0$$

$$c_c=2m\sqrt{\frac{k}{m}}=2m\,\omega_n$$

Define Damping ratio $\zeta=\frac{c}{c_c}$

$$\frac{c}{2m}=\frac{c}{c_c}\,\frac{c_c}{2m}=\zeta\,\omega_n$$

Writing w/ new terminology

$$\gamma_{1,2}=\left(-\zeta\pm\sqrt{\zeta^2-1}\right)\omega_n$$

$$x(t)=C_1e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}+C_2e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$$

Case 1: Overdamping

If $\zeta > 1$ (a.k.a $\frac{c}{2m} > \sqrt{\frac{k}{m}}$)

Then both roots are real and distinct

$$r_{1,2} = \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

Solution: $x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Behaviour: Non-oscillatory solutions, heavily damped, never pass through x_0
(No overshooting)
e.g. conference door

Case 2: Critical Damping

If $\zeta = 1$ (a.k.a $\frac{c}{2m} = \sqrt{\frac{k}{m}}$)

then square root term $= 0$ $\Rightarrow$ don't have 2 roots

$x_1 = c_1 e^{rt}$, but we need another linearly independent solution

Assertion: Reduction of order

$x_2 = c_2 t e^{rt}$

$$x(t) = (c_1 + c_2 t) e^{-\omega_n t}$$

For generic IC $(x(0) = x_0,\ \dot{x}(0) = \dot{x}_0)$

$$C_1 = x_0, \quad C_2 = \dot{x}_0 + \omega_n x_0$$

$$x_{crit} = \left[x_0 + (\dot{x}_0 + \omega_n x_0)t\right] e^{-\omega_n t}$$

Behaviour: non-oscillatory. Doesn't overshoot

returns to equilibrium (w/out overshoot) in the least possible amount of time

e.g. Shock absorbers

<u>Case 3</u>: Underdamping

If $\zeta < 1$ (a.k.a. $c < c_c$, $\frac{c}{2m} < \sqrt{\frac{k}{m}}$)

Then γ_1 and γ_2 are distinct but also imaginary

$$x(t) = C_1 e^{\left(-\zeta + i\sqrt{1-\zeta^2}\right)\omega_n t} + C_2 e^{\left(-\zeta - i\sqrt{1-\zeta^2}\right)\omega_n t}$$

Define Damped Angular Frequency, $\omega_d = \omega_n \sqrt{1-\zeta^2}$

as $\zeta \to 0$, $\omega_d \to \omega_n$

as $\zeta \to 1$, $\omega_d \to 0$

$$x(t) = C_1 e^{-(\zeta\omega_n + i\omega_d)t} + C_2 e^{-(\zeta\omega_n - i\omega_d)t}$$

$$x(t) = e^{-\zeta\omega_n t}\left[C_1 e^{-i\omega_d t} + C_2 e^{i\omega_d t}\right]$$

Need real solutions $x^* = x$

$e^{i\omega_d t}$ and $e^{-i\omega_d t}$ linearly independent

$\Rightarrow C_1^* = C_2$ and $C_2^* = C_1$

Using Euler's identity we can show

$$C_1 = \frac{A}{2} e^{i\phi_0}$$

$$x(t) = \underbrace{e^{-\zeta\omega_n t}}_{\text{exponential decay envelope}} \left[\underbrace{A\cos(\omega_d t + \phi_0)}_{\text{same as simple harmonic oscillator}}\right]$$

Lecture 5

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Assume linear viscous damping

$$m\ddot{x} + c\dot{x} + kx = 0$$ Still 2nd order homogenous

$$x = Ae^{rt} \qquad r = \frac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4\frac{k}{m}}}{2}$$

Term determines behaviour

1. 2 real → overdamped
2. 0 → critically damped
3. 2 imaginary → under damped

Examples:

Critical Damping (car suspension)

- don't overshoot, don't oscillate

If there were no damping, $T = 1s$

$$T = \frac{2\pi}{w_n}, \quad w_n = \sqrt{\frac{k}{m}}$$

After adding damper, car is displaced

$$x(0) = 0.2, \quad \dot{x}(0) = 0$$

What is the displacement from eq. @ $t=1s$

Critical damping $\Rightarrow \zeta=1 \Rightarrow \frac{c}{2m} = \sqrt{\frac{k}{m}} = \omega_n = \frac{2\pi}{T_n} \sim 1s$

$$c = 4\pi m$$

$$x(t) = \left[x_0 + (\dot{x}_0 + \omega_n x_0)t\right]e^{-\omega_n t}$$

$$x(t=1) = 0.0027\ mm$$

Underdamped

Aperiodic motion

$$\ddot{x} + \dot{x} + \pi x = 0$$

$$m=1,\ c=1,\ \pi = k$$

a) Show underdamped

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\pi}$$

Underdamped if $\zeta < 1$, $\frac{c}{2m} < \omega_n$

$\frac{1}{2} < \sqrt{\pi}$? ✓ true

b) What is damped angular frequency?

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

c) What is solution $x(t)$ given $x(0)=\pi$, $\dot{x}(0)=0$

$$x(t) = U e^{-\zeta \omega_n t} \cos[\omega_d t + \phi_0]$$

$$x(0) = \pi \Rightarrow \quad U\cos(\phi_0) =$$

$$\dot{x}(0) = 0 = -U \zeta \omega_n \cos(\phi_0) - U \omega_d \sin(\phi_0)$$

Solve for U, ϕ_0

Characteristic Time Constant

$$\tau = \frac{1}{2\zeta \omega_n}$$

$$\therefore x = U e^{-\frac{t}{2\tau}}$$

Quality Factor

- if we have weak damping,

$$Q = \frac{\sqrt{Km}}{c} = \frac{2\pi}{\Delta E / E} \qquad Q = \frac{\omega_d}{2\zeta\omega_n}$$

- $e^{-\frac{t}{\tau}} \Rightarrow$ decreases by a factor of e when $t = \tau$

Example: Middle C

$f = 255\ \text{hz} = 262\ \frac{1}{s}$

loses half of energy after 4s

what is decay time τ?

$$E = E_0 e^{-t/\tau}$$

$$\frac{E_0}{2} = E_0 e^{-4/\tau}$$

$$e^{-4/\tau} = \frac{1}{2}$$

$$-\frac{4}{\tau} = \ln\left(\frac{1}{2}\right)$$

$$\tau = \frac{-4}{\ln\left(\frac{1}{2}\right)} \approx 5.77s$$

b) Quality Factor

$$Q = \omega_n \tau = 2\pi f \tau$$

$\omega_n \approx \omega_d$ from Derivation

c) Fractional energy loss per cycle

$$\frac{T}{\tau} = \frac{1}{f\tau}$$

Logarithmic Decrement

- Amplitude from one cycle to the next

$$Ue^{-\frac{c}{2m}t} \Rightarrow Ue^{-\frac{c}{2m}\left(t+\frac{2\pi}{\omega_d}\right)}$$

X_n = envelope for n^{th} cycle

$$X_{n+1} = X_n e^{-\frac{c\pi}{m\omega_d}}$$

$$X_n \quad e^{\frac{c\pi}{n\omega_d}}$$

$$\frac{x_n}{x_{n+1}} = e^{\frac{c\pi}{m\omega_d}}$$

$$\ln\left(\frac{x_n}{x_{n+1}}\right) = \frac{c\pi}{m\omega_d} \triangleq \delta \quad \text{"logarithmic decrement"}$$

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}$$

$$\ln\left(\frac{x_n}{x_{n+1}}\right) = \delta = \frac{1}{n}\ln\left(\frac{x_n}{x_{n+1}}\right)$$

Energy Conservation:

$$E(t) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} \Rightarrow \dot{x}\underbrace{(m\ddot{x} + kx)}_{-c\dot{x}}$$

$$\frac{dE}{dt} = \underbrace{-c\dot{x}^2}_{\text{monotonically decreasing}}$$

Bus Drivers Example

- combined ζ through δ

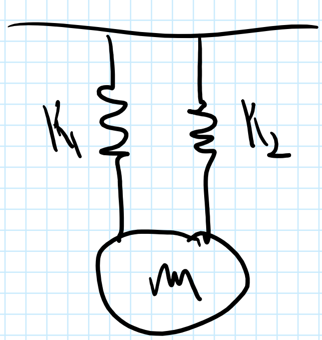
Apply these models to real system

D.O.F

- number of independent variables required to characterize model

Equivalent Springs

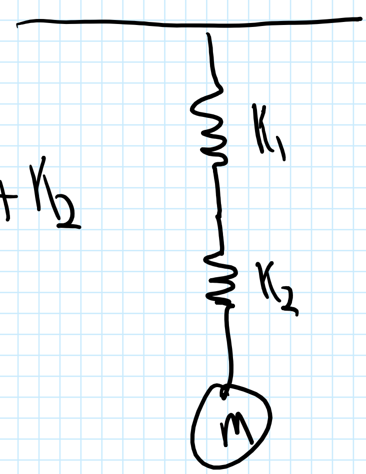
Parallel



$$K_{eq} = K_1 + K_2$$

Extension is same

Series



$$\frac{1}{K_{eq}} = \frac{1}{K_1} + \frac{1}{K_2}$$

Force is the same

Lecture 7

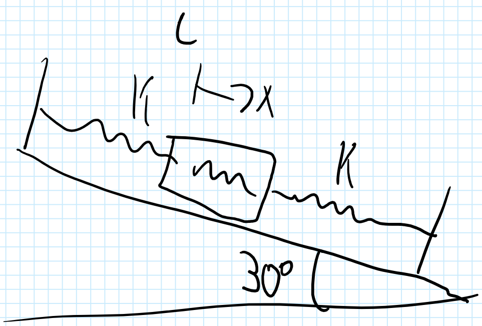
Wednesday, February 10, 2021 3:02 PM

3K

$$\frac{1}{K}+\frac{1}{3K}+\frac{1}{3K}$$

$$\left(\frac{1}{K}\left(\frac{5}{3}\right)\right)^{-1} = \frac{3}{5}K$$

$$\boxed{K_{eq} = \frac{13}{5}K}$$



a) Derive equation of motion

$$K_{eq}=2K$$

$$\sum F_i = m\ddot{x} = -2Kx - c\dot{x} + mg\sin\theta$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{K_{eq}}{m}}\sqrt{1-\left(\frac{c}{2\sqrt{mK_{eq}}}\right)^2}$$

$$\omega_d = \sqrt{\frac{2K}{\quad}}\sqrt{1-\left(\frac{c^2}{\quad}\right)}$$

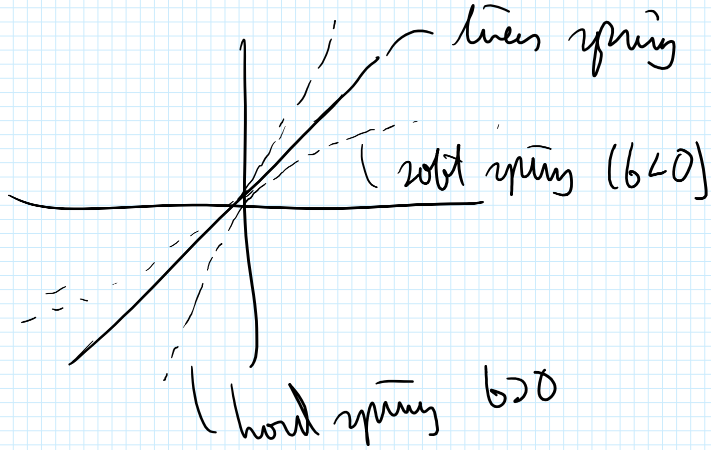
$$\omega_d = \sqrt{\frac{2k}{m}}\sqrt{1-\left(\frac{c^2}{8mk}\right)}$$

c) $\zeta < 1$ ∴ underdamped

$$x_{eq} = \frac{mg\sin\theta}{2k} = 0.05\,m$$

Spring Nonlinearity and Analogous Systems

- perfect metallic coil becomes nonlinear at sufficient deformation



Linearize non linear systems

- perform taylor expansion on force

$\Delta F = k\Delta x$, $k = \left.\frac{\partial F}{\partial x}\right|_{x=0}$ } - for small Δx's

Stainless steel system?

$$k_{eq} = \frac{\sigma A}{\epsilon L} = E\frac{A}{L}$$ } only in linear regime where E is valid

Nonlinear Systems from Linear elements

i.e. spring cart model

Generalized Coordinates and Lagrangian Method

- look at system in terms of energy landscape

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i^{(n)}$$ n d.o.f.

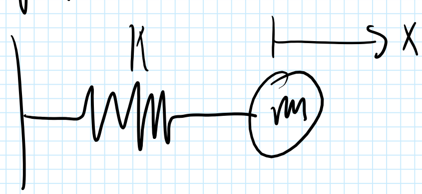
T = Total Kinetic Energy

V = Total Potential Energy

Q_i = External forces

q_i = generalized Lagrangian coordinate

Example



D.O.F = 1

$$q_1 = x$$

$$T = \frac{1}{2} m \dot{q}_1^2 = \frac{1}{2} m \dot{x}^2$$

$$V = \frac{1}{2} K q_1^2 = \frac{1}{2} K x^2$$

$$\mathcal{L} = T - V = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} K x^2$$

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0$$ No external forces

$$-Kx - (m\ddot{x}) = 0$$

$m\ddot{x} + kx = 0$

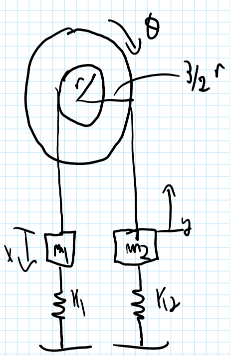
Piazza Poll

- underdamped system → Oscillatory
  - ↳ does that mean there is a period?
  - ↳ periodicity?
    - decay so never hits same values
    - define a period?

Generalized Coordinates

1. # D.O.F.
2. How do you choose your generalized coordinates

Since many terms in energy are quadratic, we can ignore signs



$m_1 = 2$
$m_2 = 1$
$K_1 = 30{,}000$
$K_2 = 10{,}000$

$$T = \frac{1}{2} m_1 (\dot{\theta} r)^2 + \frac{1}{2} m_2 \left(\dot{\theta} \frac{3}{2} r\right)^2 + \frac{1}{2} I \dot{\theta}^2$$

$$= \frac{1}{2} \underbrace{\left(m_1 r^2 + \frac{9}{4} r^2 m_2 + I\right)}_{m'} \dot{\theta}^2$$

$$V = \frac{1}{2} K_1 x^2 + \frac{1}{2} K_2 y^2 + \cancel{\Delta \text{ gravitational}}$$

→ neglect because spring dominates

$$V = \frac{1}{2} K_1 r^2 \theta^2 + \frac{1}{2} K_2 \left(\frac{3}{2} r \theta\right)^2$$

$$V = \frac{1}{2} \underbrace{\left(K_1 r^2 + \frac{9}{4} K_2 r^2\right)}_{K'} \theta^2$$

$$\mathcal{L} = T - V = \frac{1}{2}\left(m_1 r^2 + \frac{9}{4} m_2 r^2 + I\right)\dot{\theta}^2 - \frac{1}{2}\left(K_1 + \frac{9}{4} K_2\right) r^2 \theta^2$$

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right)$$

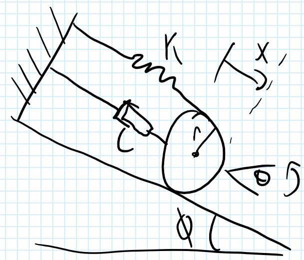
$$-\left(K_1 + \frac{9}{4} K_2\right) r^2 \theta - \left(m_1 r^2 + \frac{9}{4} m_2 r^2 + I\right) \ddot{\theta} = 0$$

$$m_{eq}\ddot{\theta} + K_{eq}\theta = 0$$

$$\theta(t) = \theta_0 \sin(\omega_n t + \phi_0)$$

$$\omega_n = \sqrt{\frac{K_{eq}}{m_{eq}}}$$

Example #2



$$T = \underbrace{\frac{1}{2}I\dot{\theta}^2}_{\text{rot}} + \underbrace{\frac{1}{2}m(r\dot{\theta})}_{\text{trans C.O.M}}$$

$$V = \underbrace{\frac{1}{2}K(2\theta r)^2}_{\text{spring potential}} + \underbrace{mg \sin(\phi) r\theta}_{\text{gravitational potential}}$$

$$= 2Kr^2\theta^2 + mg\sin(\phi)r\theta$$

$$Q = -c\dot{x} = cr\dot{\theta}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

$$\frac{\partial}{\partial t}\left(\left(I + mr^2\right)\dot{\theta}\right) - 4Kr^2\theta + mgr\sin\phi = cr\dot{\theta}$$

$$\underbrace{\left(I + mr^2\right)}_{m_{eq}}\ddot{\theta} - \underbrace{cr}_{c_{eq}}\dot{\theta} - \underbrace{4Kr^2}_{K_{eq}}\theta = -mgr\sin(\phi)$$

Assume underdamped: $\theta(t) = Ue^{-\zeta\omega_n t}\cos(\omega_d t + \phi_0)$

m_{eq}, c_{eq}, K_{eq} ζ - known

$$\omega_n = \sqrt{\frac{K_{eq}}{m_{eq}}}, \quad \zeta = \frac{c_{eq}}{2\sqrt{K_{eq}m_{eq}}}, \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

Coulombic Friction

- how is this similar/ different to linear viscous damping?

$F = \mu m g$, always acting against direction of motion

Newton's 2nd

$$\Sigma F = -Kx - \mu m g \frac{\dot{x}}{|\dot{x}|}$$

$$m\ddot{x} + \mu m g \frac{\dot{x}}{|\dot{x}|} + Kx = 0$$ to avoid abs. value split into 2 ODEs

$$m\ddot{x} + \mu m g + Kx = 0$$ Moving Right

$$m\ddot{x} - \mu m g + Kx = 0$$ Moving Left

$$x(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t) - \frac{\mu m g}{K}, \quad \dot{x} > 0$$

$$x(t) = C_3 \cos(\omega_n t) + C_4 \sin(\omega_n t) + \frac{\mu m g}{K}, \quad \dot{x} < 0$$

Start w/ I.C.

$x(0) = x_0 > 0$ $\qquad$ $\dot{x}(0) = 0$

$\therefore$ going to move left

$x(t=0)$ $\quad C_3 = x_0 - \frac{\mu m g}{K}$

$\dot{x}(t=0) = 0 \Rightarrow C_4 = 0$

$$x(t) = \left(x_0 - \frac{\mu m g}{K}\right)\cos(\omega_n t) + \frac{\mu m g}{K}$$

Only valid until sign of $\dot{x}$ flips

This occurs @ $\quad T = T_1 = \frac{\pi}{\omega_n} = \frac{\text{Period}}{2}$

$$x_1 = x(T_1) = -\left(x_0 - \frac{2\mu m g}{K}\right)$$

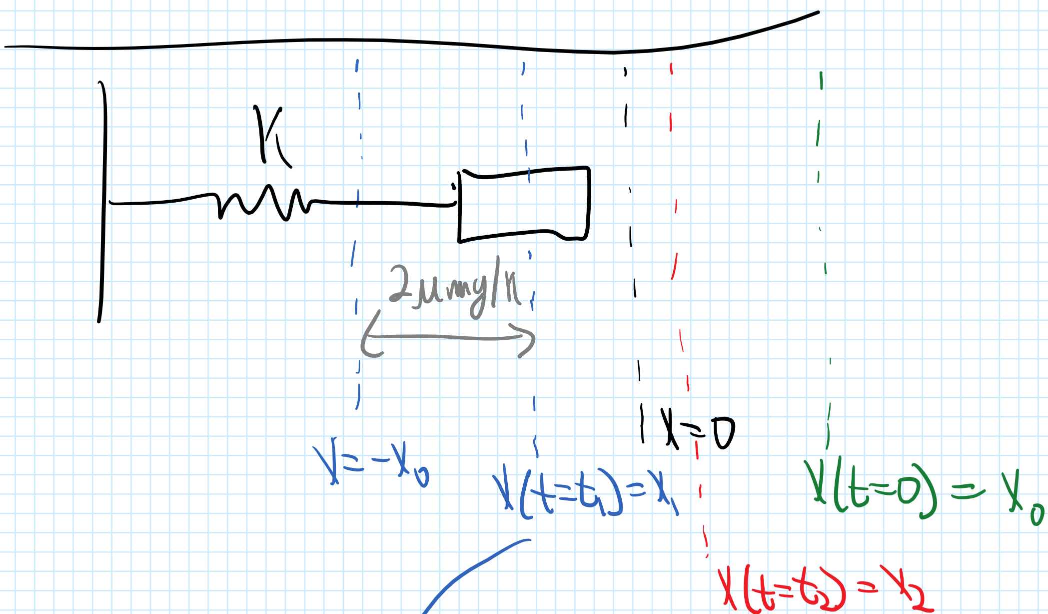
Lecture 9

Wednesday, February 17, 2021 3:09 PM

2 Solutions $x(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t) \pm \frac{\mu m g}{K}$

$\dot{x} < 0$

$\dot{x} > 0$



$$x_1 = -x_0 + \frac{2\mu m g}{K}$$

From $t = t_1 \to t_2$ $\dot{x} > 0$

$$x(t) = C_1 \cos(\omega_n(t-t_1)) + C_2 \sin(\omega_n(t-t_1)) - \frac{\mu m g}{K}$$

Apply new "IC"

$$x(t_1) = x_1 \ , \ \dot{x}(t_1) = 0$$

$$x_1 = C_1 \cos(0) + \cancel{C_2 \sin(0)}^{\,0} - \frac{\mu m g}{K}$$

$$x_1 = C_1 \cos(0) + \cancel{C_2 \sin(0)} - \frac{\mu m g}{K}$$

$$C_1 = x_1 + \frac{\mu m g}{K_1}$$

$$\Rightarrow C_2 = 0$$

$$x(t) = \left(x_1 + \frac{\mu m g}{K_1}\right)\cos\left(\omega_n (t - t_1)\right) - \frac{\mu m g}{K_1}$$

$$t_2 = t_1 + \frac{\pi}{\omega_n}$$

$$x(t_2) = -x_1 - \frac{2\mu m g}{K}$$

$$= x_0 - \frac{4\mu m g}{K}$$

Now we lose a fixed value, envelope will look different

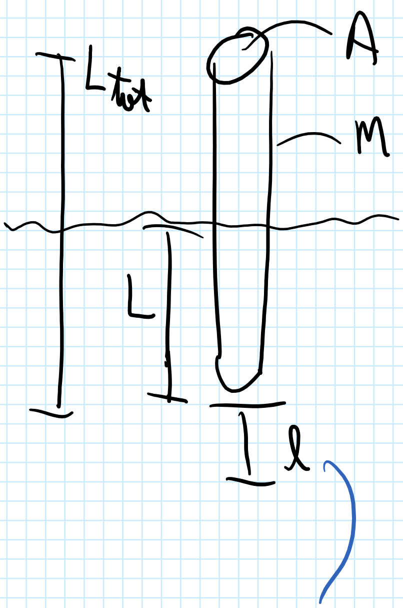
- linear decay envelope
- not exponential like linear viscous damping

Other types of damping

- Thermal / Heating
- Hysteresis

- Hysteresis
  - produce hysteresis loop under repeated loading
    - could include material damage
    - fluid flow into foam pores
    - etc

Can we model realistic systems in terms of our simple model



Find Eq Position

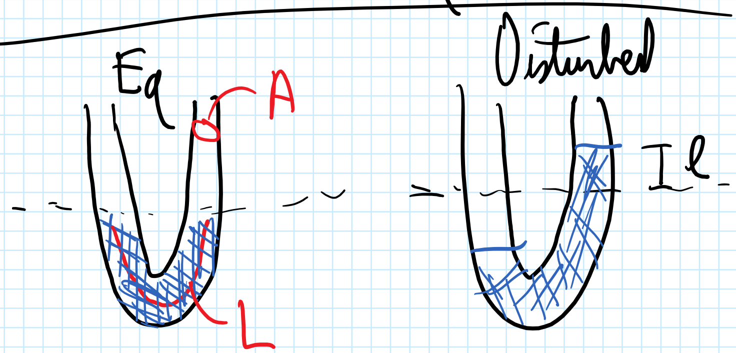
(gravitational force = buoyancy force)

$$mg = -F_B = -A \cdot L \rho_{water} \cdot g$$

$$L = \frac{m}{A\rho}$$

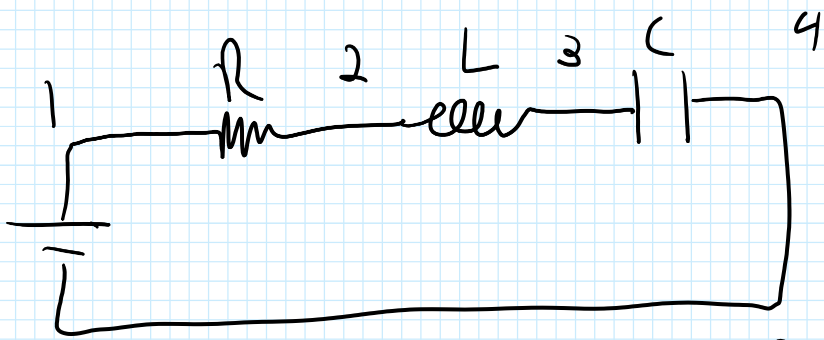
$$\omega_n = \sqrt{\frac{K_{eq}}{m_{eq}}}$$

F Disturbed



$$m_{eq} = AL\rho$$

$$F = -mg = -2lA\rho g$$



Q = charge

I = current

$$V_1 - V_2 = RI = R\dot{Q}$$

$$V_2 - V_3 = L\dot{I}$$

$$V_3 - V_4 = \frac{Q}{C}$$

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = 0$$

m_{eq} $\quad$ c_{eq} $\quad$ K_{eq}

m_{eq} $\quad$ c_{eq} $\quad$ K_{eq}

Lecture 10

Monday, February 22, 2021 3:06 PM

Example Dynamics of Molecules

$$U = \epsilon\left[\underbrace{\left(\frac{r_0}{r}\right)^{12}}_{\text{repulsive energy}} - 2\underbrace{\left(\frac{r_0}{r}\right)^{6}}_{\text{attractive energy}}\right]$$

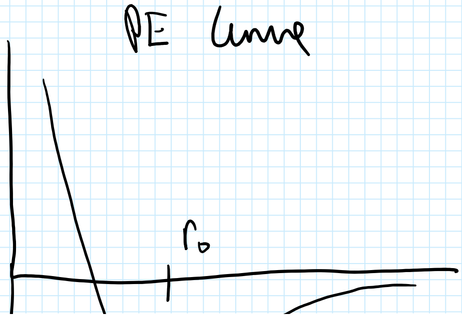
$$\frac{dU}{dr} = \epsilon\left[-12\frac{r_0^{12}}{r^{13}} + 12\frac{r_0^{6}}{r^{7}}\right]$$

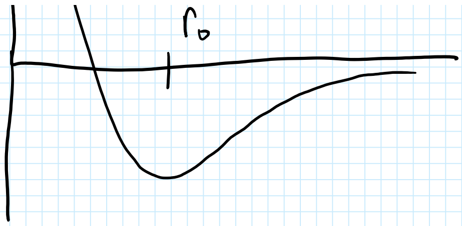
find minimum $\Rightarrow$ set this to 0

$$\frac{-12}{\epsilon r}\underbrace{\left[\frac{r_0^{12}}{r^{12}} - \frac{r_0^{6}}{r^{6}}\right]}_{=0} = 0$$

$$\frac{r_0^6}{r^6} = 1 \Rightarrow r_0 = r$$

PE Curve





$$U(r=r_0) = \epsilon\,(1-2) = -\epsilon$$

linearize about r_0

$$U(r) = U(r_0) + \frac{dU}{dr}\Big|_{r_0}(r-r_0) + \frac{1}{2}\frac{d^2U}{dr^2}\Big|_{r_0}(r-r_0) + \cdots$$

$$= -\epsilon + \frac{1}{2}\underbrace{\frac{d^2U}{dr^2}\Big|_{r_0}}_{k_{eq}}(r-r_0)^2$$

$$\frac{d^2U}{dr^2} = \frac{12\epsilon}{r^2}\left((-13)\frac{r_0^{13}}{r^{14}} - 7\left(\frac{r_0}{r}\right)^6\right)$$

$$\frac{d^2U}{dr^2}\Big|_{r_0} = \frac{72\epsilon}{r_0^2} = k_{eq}$$

m_{eq} is the reduced mass

$$m_{eq} = \mu = \frac{m_1 \cdot m_2}{m_1 + m_2}$$

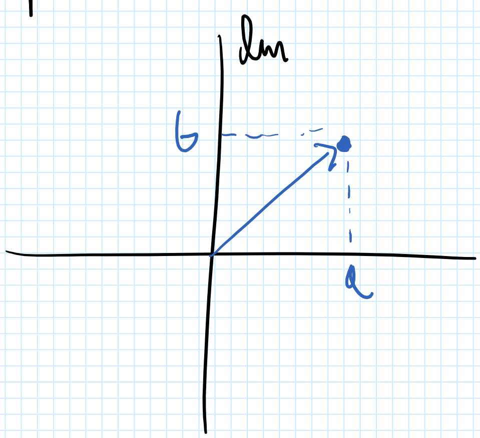
Stability in Complex Plane

$$x = Ae^{rt}$$

$$r_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

3 cases $\Rightarrow$ 3 different solutions

Complex Plane



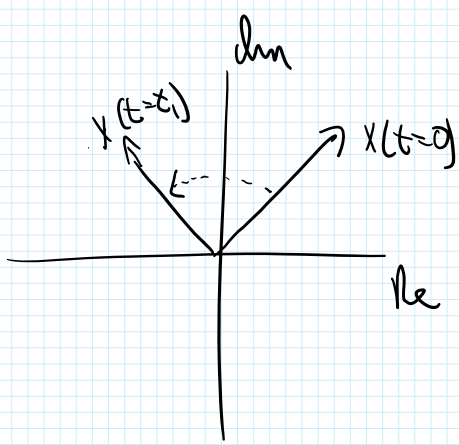
$x = a + ib$

$x = A\cos\theta + i A\sin\theta$

$A = \sqrt{a^2+b^2}$

$\theta = \operatorname{atan}\left(\frac{b}{a}\right)$

Harmonic function on complex plane



$$x = Ae^{i\theta} = Ae^{i\omega t}$$

$$\dot{x} = i\omega x$$

$$\ddot{x} = -\omega^2 x$$

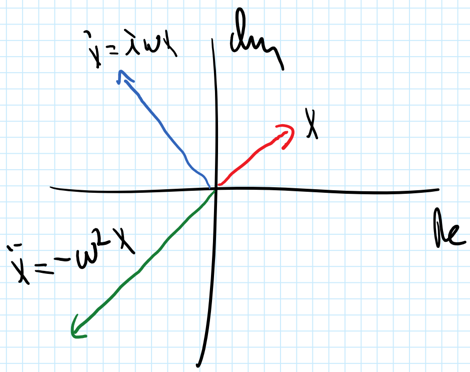
To get physical quantities we need real components

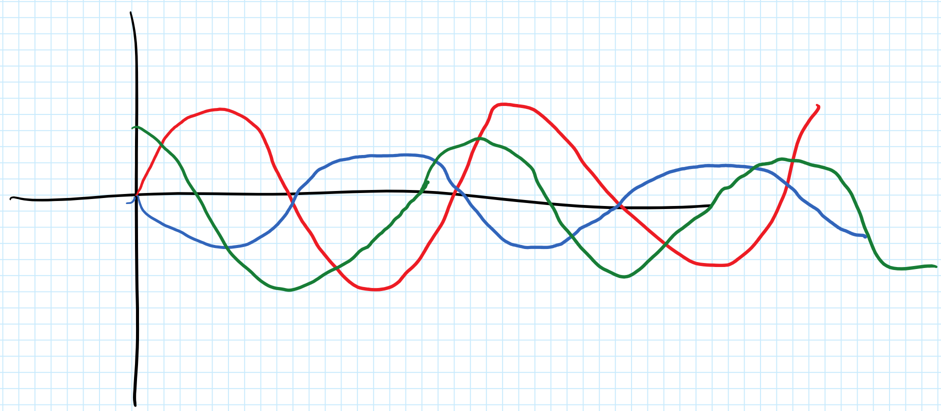
Displacement $= \mathrm{Re}[x] = A\cos\omega t$

Velocity $= \mathrm{Re}[\dot{x}] = \omega A \sin(\omega t + \pi)$

$$= \omega A \cos\left(\omega t + \frac{\pi}{2}\right)$$

Acceleration $= \omega^2 A \cos(\omega t + \pi)$





Stability

- use complex plane to analyze

$$r_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

$$x = a + ib = A\cos\theta + iA\sin\theta$$

$$= Ae^{i\theta}$$

Position, velocity, acceleration lie in $\perp$ orientation

- rotate w/ omega

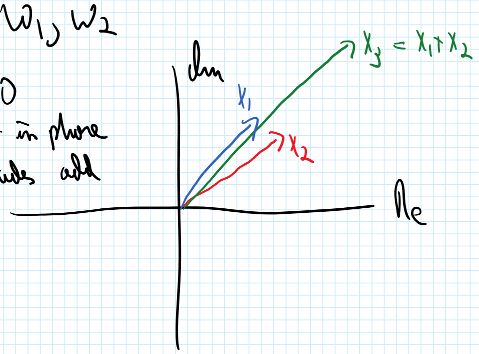
What if rotations aren't exactly in sync?

Beats

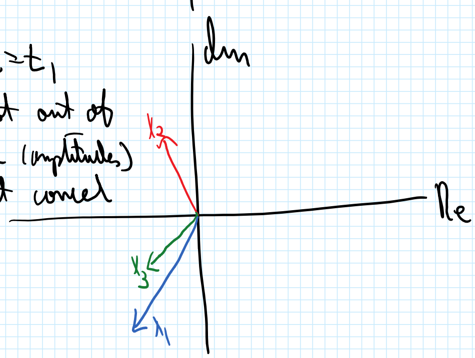
two functions w/ almost the same frequencies

ω_1, ω_2

$t = 0$
almost in phase
amplitudes add



$t = t_1$
almost out of phase (amplitudes) almost cancel



example:

$$x_1 = 8\cos(10t) = \mathrm{Re}\left[X_1 = 8e^{10it}\right]$$

$$x_2 = 11\cos(11t) = \mathrm{Re}\left[X_2 = 11e^{11it}\right]$$

at $t = 1$ second

X_1 has changed phase by 10 rad

X_2 has changed phase by 11 rad

relative change $= 1$ rad

after 2π seconds, relative phase is same as $t = 0$

every π seconds goes from max → min

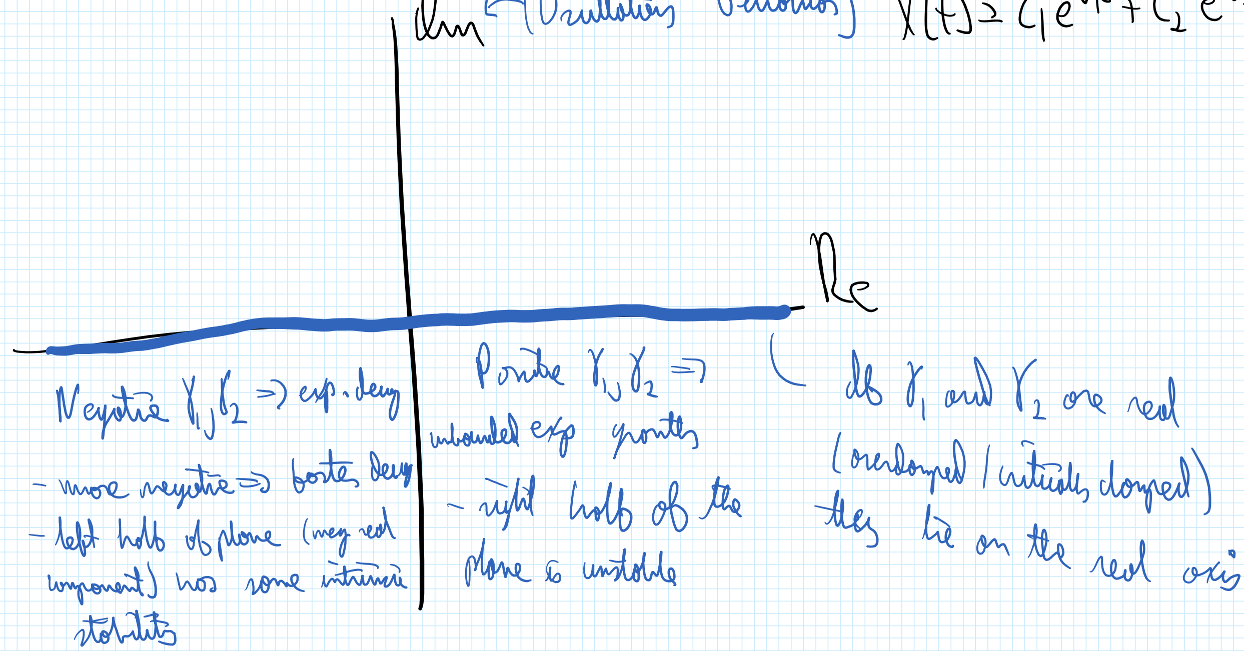
$$T_{beat} = 2\pi \text{ seconds}$$

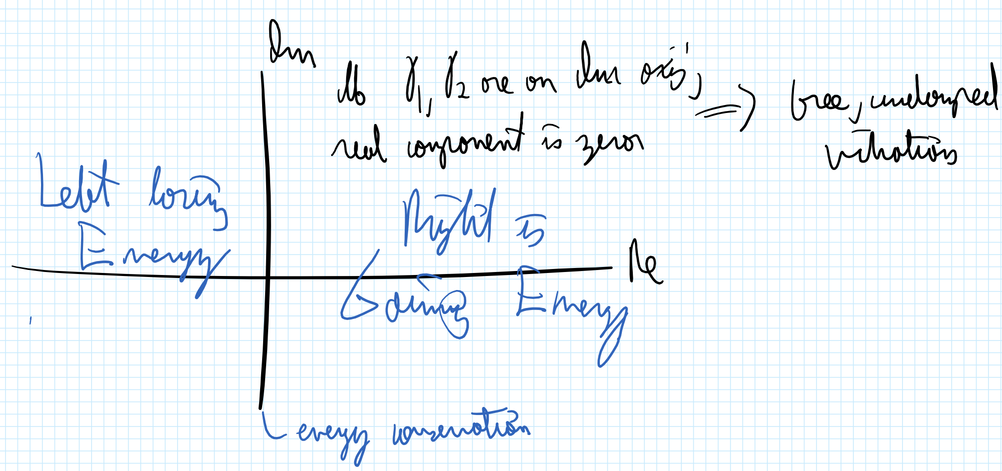
$$f_{beat} = \frac{1}{T_{beat}} = 0.16\,Hz$$

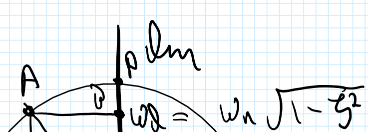
What are max and minimum amplitude

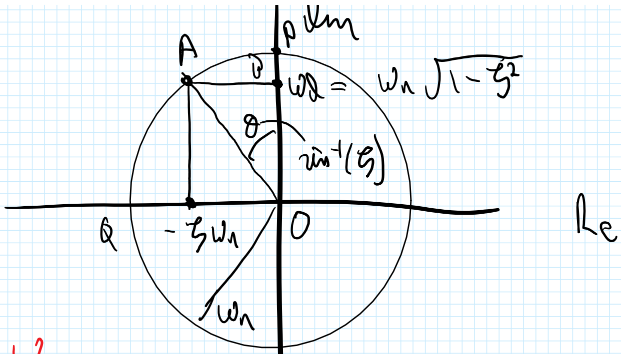
- max is when totally in phase ⇒ add amplitudes $8+11=19$
- minimum is when totally opposite subtract $11-8=3$

$$x(t) = C_1 e^{\gamma_1 t} + C_2 e^{\gamma_2 t}$$



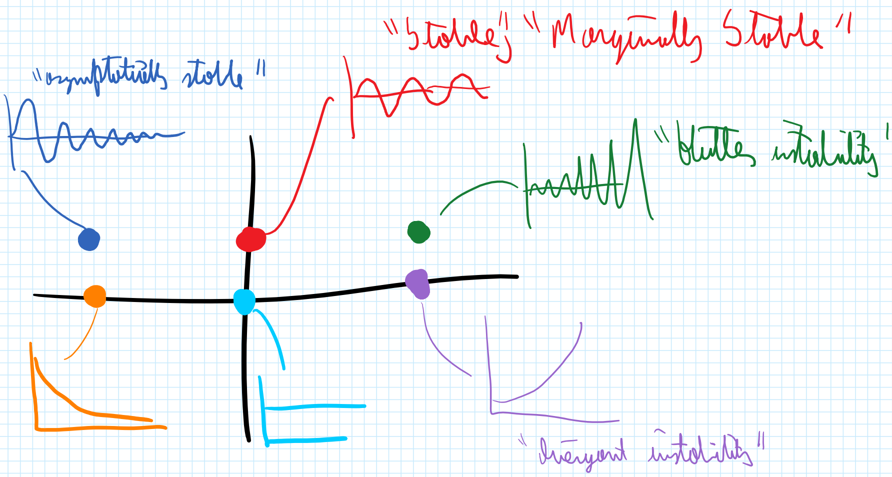




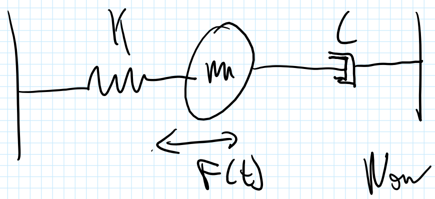


- Radius = ω_n
- Anything along semicircle has same ω_n
- Different angles θ mean different amounts of damping

- different $\omega_n \Rightarrow$ different radii



Forced Vibrations
- equations of motion



Now we add a force to the system

$$\Sigma F = m\ddot{x} = -Kx - c\dot{x} + F(t)$$

$$m\ddot{x} + c\dot{x} + Kx = F(t)$$

Non-homogeneous ODE

What is F(t)?

1. F(t) is periodic and harmonic — sin or cos of single frequency
   strategy: Solve directly
2. F(t) is periodic but not harmonic
   Strategy: Periodic functions can be expanded into sum of harmonic functions of different frequencies. (Via Fourier Series)
3. F(t) is not periodic
   Strategy: ways to do this but depends on specifics

Case 1 F(t) is harmonic

$$m\ddot{x} + c\dot{x} + Kx = F_0 e^{i(\omega t + \phi)}$$

freq of oscillation
phase offset
amplitude of oscillation force

First, we already know homogeneous solution

i.e. $m\ddot{x}_h + c\dot{x}_h + Kx_h = 0$

Now we need x_p, particular solution

$$x = x_h + x_p$$

We know if $c>0$, @ $t \to 0$, $x_h \to 0$

$\therefore$ x_p becomes the steady state solution

x_h is the transient solution

How to find x_p?

Lets start w/ $c=0$

$$m\ddot{x} + kx = F_0 \cos(\omega t)$$

$$x_h = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t)$$

For x_p, lets assume $x_p(t) = A\cos(\omega t + \phi)$

$$\dot{x}_p = -A\omega \sin(\omega t + \phi)$$

$$\ddot{x}_p = -A\omega^2 \cos(\omega t + \phi) = -\omega^2 x$$

Plug these terms into above ODE

$$-A\omega^2 m \cos(\omega t + \phi) + kA\cos(\omega_n t + \phi) = F_0 \cos(\omega t)$$

True if $\phi = 0$ or π

If $\phi = 0$ $\quad -A\omega^2 m + kA = F_0$

$$A = \frac{F_0}{k - \omega^2 m} = \frac{F_0/k}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

This is for $\phi = 0$ and $\omega < \omega_n$ (Physically, A is positive)
Low frequency case

If $\phi = \pi$ we get a negative sign

$$A = \frac{F_0}{\omega^2 m - K} = \frac{F_0/K}{\left(\frac{\omega}{\omega_n}\right)^2 - 1}$$

when $\phi = \pi$

$\omega > \omega_n$

High Freq. case

We can now write the general solution as

$$x(t) = x_h + x_p = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t) + \frac{F_0}{K - \omega^2 m}\cos(\omega t)$$

Magnification Factor

Look again at expression for K

$$A = \frac{F_0/K}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

static displacement

$$\delta_{st} = \frac{F_0}{K}$$

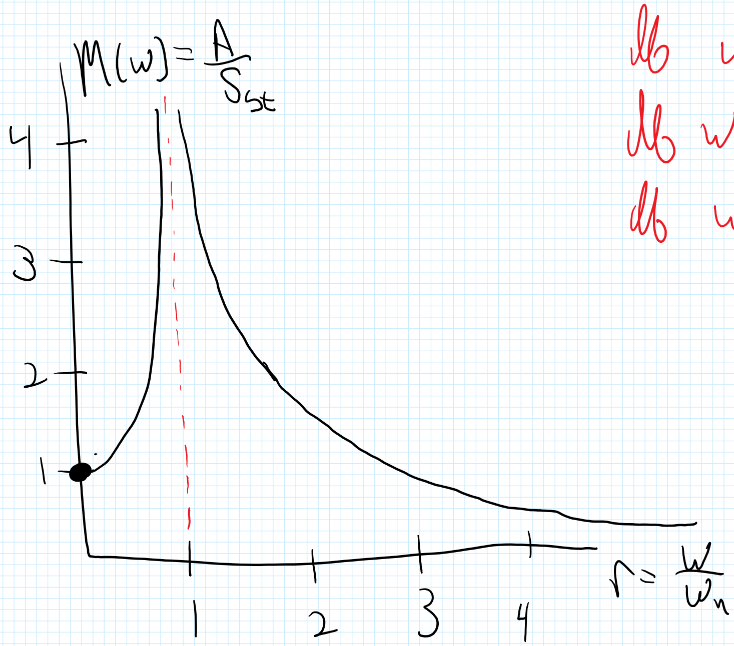
Hooke's Law: $F = -Kx \Rightarrow -x = \frac{F_0}{K}$

$$\Rightarrow \quad A = \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \qquad \frac{A}{\delta_{st}} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \rightarrow \text{unit less}$$

Magnification factor, M

$M(\omega)$

$$M \triangleq \frac{A}{\delta_{st}}$$



If $\omega = 0$, $\rightarrow$ Static Force $\rightarrow M = 1$

If $\omega = \omega_n \rightarrow M(\omega) \rightarrow \infty$

If $\omega \rightarrow \infty$, $\rightarrow M(\omega) \rightarrow 0$

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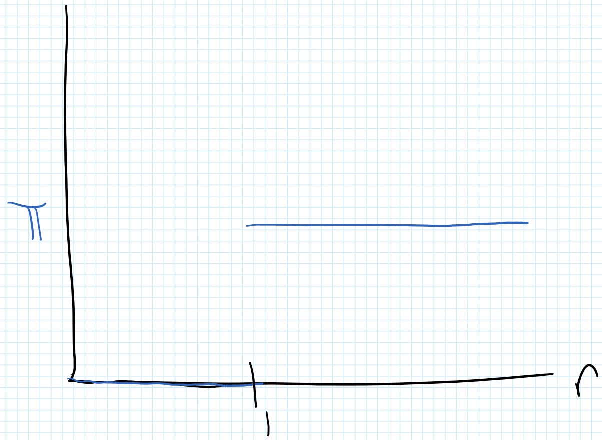
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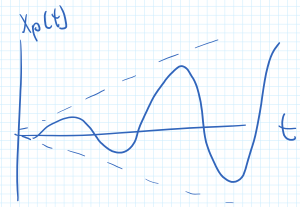
What happens to $\omega = \omega_n$ in an undamped system?

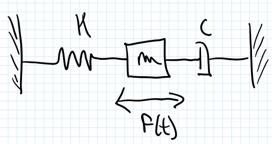
$$X(t) = X_h + X_p = C_1 \cos \omega_n t + C_2 \sin \omega_n t + \frac{F_0}{K - \omega^2 m} \cos(\omega t)$$

$$= C_1 \cos \omega_n t + C_2 \sin \omega_n t + \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \cos \omega t$$

$$\lim_{\omega \to \omega_n} \left[\frac{\delta_{st} \cos(\omega t)}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \right] = \lim_{\omega \to \omega_n} \left(\frac{\frac{d}{d\omega}\left(\delta_{st} \cos(\omega t)\right)}{\frac{d}{d\omega}\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)} \right) = \lim_{\omega \to \omega_n} \left(\frac{-t\delta_{st} \sin(\omega t)}{-2\frac{\omega}{\omega_n^2}} \right)$$

$$= \frac{t\delta_{st} \sin(\omega_n t)}{\frac{2}{\omega_n}} = \underbrace{\frac{\omega_n t \delta_{st}}{2}}_{\text{(envelope w/ linear increase}} \sin(\omega_n t)$$





$$m\ddot{x} + c\dot{x} + Kx = F_0 e^{i\omega t}$$

Trial Solution

$$x_p = Ae^{i(\omega t - \phi)}$$

$$\dot{x}_p = i\omega A e^{i(\omega t - \phi)}$$

$$\ddot{x}_p = -\omega^2 e^{i(\omega t - \phi)}$$

$$-m\omega^2 A e^{i(\omega t - \phi)} + i\omega C A e^{i(\omega t - \phi)} + KA e^{i(\omega t - \phi)} = F_0 e^{i\omega t}$$

$$-m\omega^2 A + i\omega C A + KA = F_0 e^{i\omega t} \cdot e^{-i\omega t} e^{i\phi}$$

$$A(-m\omega^2 + i\omega C + K) = F_0 e^{i\phi}$$

Two Separate Equations by equating real and imaginary components

Real (I) $-m\omega^2 A + KA = F_0 \cos(\phi)$

Im (II) $\omega c A = F_0 \sin(\phi)$

$\frac{II}{I}$: $\tan(\phi) = \dfrac{\omega c}{K - m\omega^2}$

$$\phi = \tan^{-1}\left(\frac{\omega c}{K\left(-\frac{m}{K}\omega^2 + 1\right)}\right)$$

Recall $\zeta = \dfrac{c}{2m\omega_n}$

$$= \tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)$$

Recall $r = \dfrac{\omega}{\omega_n}$

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$

$I^2 + II^2$

$$\Rightarrow \quad A^2(-m\omega^2 + K)^2 + A^2(\omega c)^2 = F_0^2\underbrace{(\sin^2 + \cos^2)}_{1}$$

$$A = \frac{F_0}{\sqrt{(K - m\omega^2)^2 + (\omega c)^2}}$$

As $\omega \to 0$, $A = \dfrac{F_0}{K} = \delta_{st}$

$\omega \to \infty$, $A \to \dfrac{F_0}{m\omega^2}$

$$M = \frac{A}{\delta_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

What about limit as $\omega \to \omega_n$?

$$M \to \frac{1}{2\zeta}$$

As long as $\zeta > 0$, we have a steady state amplitude
— no divergence

For what value of ω will $A(\omega)$ be largest?

$$\frac{dA}{d\omega} = 0 = \frac{d}{d\omega}\left(\frac{\delta_{st}}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}\right)$$

$$= -\frac{1}{2}\delta_{st}\left(\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right) + \left(2\zeta\frac{\omega}{\omega_n}\right)^2\right)^{-3/2}\left(2\left(1-\frac{\omega}{\omega_n}\right)^2 \cdot -2\frac{\omega}{\omega_n}\cdot\frac{1}{\omega_n} + 2\left(2\zeta\frac{\omega}{\omega_n}\right)\frac{2\zeta}{\omega_n}\right)$$

very similar to ω_d

$$\omega = \omega_n\sqrt{1-2\zeta^2} \triangleq \omega_r$$

As $\zeta \to 0$, $\omega_r \approx \omega_d \approx \omega_n$

Harmonic Forcing of Damped Systems

$$m\ddot{x} + c\dot{x} + kx = F_0 e^{i\omega t}$$

$$x_p = A e^{i(\omega t - \phi)}$$

$$\phi = \tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right) = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$

$$A = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (\omega c)^2}}$$

Magnification Factor

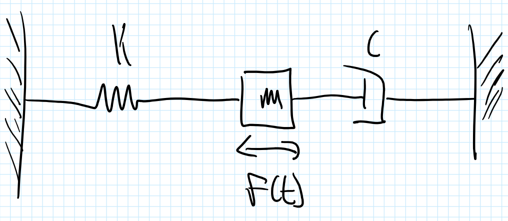
$$\frac{A}{\delta_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Peak Amplitude

$$\frac{\partial A}{\partial \omega} = 0 \Rightarrow \omega_r = \omega_n\sqrt{1-2\zeta^2}$$

as $\zeta \to 0$, $\omega_r = \omega_n = \omega_d$

Example



$m = 100$ kg
$k = 10^6$ N/m
$\zeta = 0.1$
$F_0 = 10^4$ N,
$f = 7$ hz $\Rightarrow \omega = 2\pi 7 = 44$ rad/s

What is position of mass at $t = 10$ s

$$x(t=0) = 0.1 \text{ m}, \quad \dot{x}(t=0) = 0 \frac{m}{s}$$

First we know this is underdamped

$$\Rightarrow \qquad x_h = Ue^{-\zeta\omega_n t}\cos(\omega_d t + \theta_0)$$

$$\omega_n = \sqrt{\frac{k}{m}} = 31.6 \qquad \omega_d = \omega_n\sqrt{1-\zeta^2} =$$

$$r = \frac{\omega}{\omega_n} = 1.39 \qquad M = \frac{A(\omega)}{\delta_{st}} = 1.03$$

$$A = 1.03 \cdot \delta_{st} = 1.03\frac{F_0}{k} = 0.103$$

$$\phi = \tan^{-1}\left[\frac{2\zeta r}{1-r^2}\right]$$

$$x = x_h + x_p$$

$$x = Ue^{-\zeta\omega_n t}\cos(\omega_d t + \theta_0) + 0.103e^{i(\omega t + 0.289)}$$

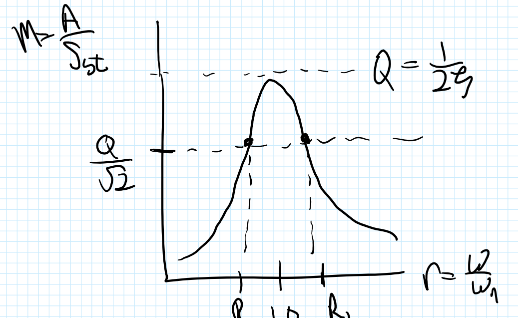
Apply IC's $\quad x(0) = \ldots \quad \dot{x}(0) = \ldots \quad \Rightarrow U = , \ \theta_0 = \ldots$

By 10s, transient solution ≈ 0

$\therefore$ Really only care about particular solution

$$x(t=10) = x_p(t=10) = \ldots$$

"Quality Factor" and width of resonance



$$E \sim A^2 \Rightarrow \left(\frac{Q}{\sqrt{2}}\right)^2 = \frac{Q^2}{2}$$

R_1, R_2 are "half power points"

$$M = \frac{A}{\delta_{st}} = \frac{Q}{\sqrt{2}}$$

$$r^4 - r^2(2-4\zeta^2) + (1-8\zeta^2) = 0$$

$\hookrightarrow$ look at roots

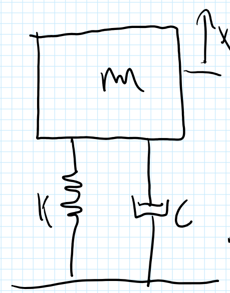
$$R_1^2 \approx \left(\frac{\omega_1}{\omega_n}\right)^2 = 1 - 2\zeta$$

$$R_2^2 \approx \left(\frac{\omega_2}{\omega_n}\right)^2 = 1 + 2\zeta$$

$$\Delta\omega = \omega_2 - \omega_1 \approx 2\zeta\omega_n$$

$$Q \approx \frac{1}{2\zeta} \approx \frac{\omega_n}{\omega_2 - \omega_1}$$

Excitation of The Base and Transmissibility



$$\underbrace{y(t) = Y_0 \sin(\omega t)}_{\text{(Displacement control not force}}$$

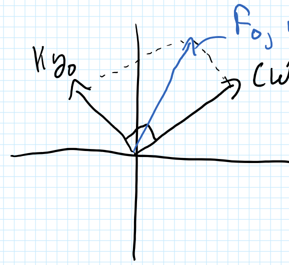
ODE:

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$

$$m\ddot{x} + c\dot{x} + kx = ky + c\dot{y}$$

$$m\ddot{x} + c\dot{x} + kx = kY_0 \sin(\omega t) + c\omega Y_0 \cos(\omega t)$$

F_0, write as sum of sin and cos w/ offset



$$\therefore \; m\ddot{x} + c\dot{x} + kx = F_0 \sin(\omega t - \alpha)$$

$\therefore \quad m\ddot{x} + c\dot{x} + kx = F_0 \sin(\omega t - \alpha)$

$$F_0 = Y_0\sqrt{k^2 + (c\omega)^2} \qquad \alpha = \tan^{-1}\left(\frac{-c\omega}{k}\right)$$

$$x_p(t) = \frac{Y_0\sqrt{k^2 + (c\omega)^2}}{\left[(k - m\omega^2)^2 + (c\omega)^2\right]^{1/2}} \sin(\omega t - \phi_1 - \alpha)$$

Manipulate + Simplify $\Rightarrow$

$$x_p(t) = A \sin(\omega t - \phi)$$

$$\phi = \tan^{-1}\left(\frac{mc\omega^3}{k(k - m\omega^2) + (\omega c)^2}\right) = \tan^{-1}\left(\frac{2\zeta r^3}{1 + (4\zeta^2 - 1)r^2}\right)$$

Define "displacement transmissibility"

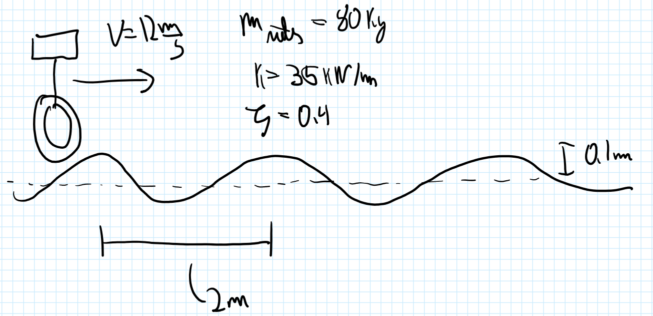
$$T_d = \frac{A}{Y_0} = \left[\frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2}\right]^{1/2} = \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}\right]^{1/2}$$

Rotating Unbalance

eccentricity

$$A = \frac{me\omega^2}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}}, \quad \phi = \tan^{-1}\left(\frac{c\omega}{k - M\omega^2}\right) = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$$

$$\frac{MA}{me} = \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$



How much will rider displace in S-S

$$\omega = 2\pi f = 2\pi \frac{12 m/s}{2m} = 37.7 \frac{rad}{s}$$

$$\omega_n = \sqrt{k/m} = 20.9\, rad/s$$

$$r = \frac{\omega}{\omega_n} = 1.8$$

$$T_d = \left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right]^{1/2} = 0.66$$

$$\text{Displacement} = Y_0 \cdot T_d = 6.6\, cm$$

Periodic But Not Harmonic?

- Apply Fourier Transform
- break F into sum of harmonic terms
- find each particular solution and sum them

Fourier's Theorem

the periodic function $F(t)$ can be expanded as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right], \quad n \in \mathbb{Z}$$

Find $a_0, a_1, b_1, \ldots a_n, b_n$

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\, dt$$

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)\,dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)\,dt$$

Background Info

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega t) + b_n\sin(n\omega t)\right]$$

multiply both sides by $\cos(m\omega t)$ and integrate over 1 period

$$\int_{-\pi/\omega}^{\pi/\omega} f(t)\cos(m\omega t)\,dt = \frac{a_0}{2}\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}}\cos(m\omega t)\,dt + \sum\left[a_n\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}}\cos(m\omega t)\cos(n\omega t)\,dt + \int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} b_n\cos(m\omega t)\sin(m\omega t)\,dt\right]$$

Orthogonality

$\int_a^b g(x)h(x)\,dx$ If g and h are orthogonal over interval $[a,b]$ iff

$$\int_a^b g(x)\,h(x)\,dx = 0$$

Sines and Cosines are $\perp$ according to these rules

Odd functions

$$H(-t) = -H(t)$$

Even Functions

$$G(-t) = G(t)$$

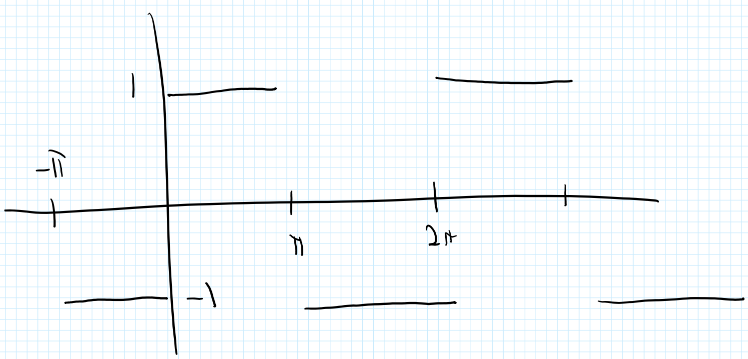
For odd functions even cosines contribute nothing

"Fourier Sine Series"

For even functions odd sine terms contribute nothing

"Fourier Cosines Series"

Example Odd Square Wave



Periodic extension

- odd vs even extension

Recall early in our course we used 2 equivalent forms for general solution

$$x = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

$$x = A \cos(\omega t - \phi)$$

Similarly we can write Fourier series w/ second notation

$$x(t) = d_0 + d_1 \cos(\omega t - \phi_1) + d_2 \cos(2\omega t - \phi_2) + \dots$$

$$d_0 = \frac{a_0}{2} \quad d_n = (a_n^2 + b_n^2)^{1/2} \quad \phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$$

Complex Fourier Notation

$$e^{i\omega t} = \cos(\omega t) + i \sin \omega t$$

$$\cos(\omega t) = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

$$\sin(\omega t) = \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{inwt}$$

$$c_n = \frac{1}{T} \int_0^T f(t) e^{-iwnt} \, dT$$

Time domain vs. frequency domain

- frequency on x axis w/ amplitude on y axis

Example

$T = 2\pi$ s, $F_0 = 3000$, $m = 4000$, $K = 64 \cdot 10^3$, $\zeta = 0.1$

$$f(t) = F_0 \cdot \frac{4}{\pi} \left(\cos(t) - \frac{\cos 3t}{3} + \frac{\cos 5t}{5} - \frac{\cos 7t}{7} \right)$$

$$w_n = \sqrt{\frac{K}{m}} = 4 \text{ rad/s}$$

4 forcing terms $m = 1, 3, 5, 7$

$w =$

$r_1 = \frac{w_1}{w_n} = 0.25$

$r_2 = 0.75$

$r_3 = 1.25$

$r_4 = 1.75$

$$x_p = \sum_{1}^{4} x_m$$

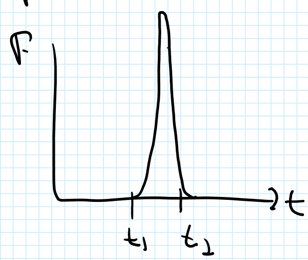
$$x_m = A_m e^{i(w_n t - \phi)}$$

$$\phi_m = \tan^{-1}\left(\frac{2\zeta r_m}{1 - r_m^2}\right)$$

$$A_m = \frac{\delta_{st,m}}{\sqrt{(1 - r_m^2)^2 + (2\zeta r_m)^2}} \qquad \frac{F_{0,m}}{k}$$

3. $F(t)$ is not periodic

Impulses



Impulse

$$I = \int_{t_1}^{t_2} F(\tau)\, d\tau$$

$$[I] = N\cdot s = kg\,\frac{m}{s}$$

$$I = \int_{t_1}^{t_2} F(\tau)\, d\tau = \Delta p = m\Delta v$$

$$\Delta v = \frac{I}{m}$$

I sets initial conditions. After, system evolves like x_h

Assume $\zeta < 1 \Rightarrow$ Underdamping

$$x = C e^{-\zeta\omega_n t}\cos(\omega_d t + \phi_0)$$

$$x(0) = 0 = C\cos(\phi_0) \to \phi_0 = \frac{\pi}{2} + n\pi = \frac{\pi}{2}(2n+1)$$

$$\dot{x}(0) = \frac{I}{m} = -\zeta\omega_n C \overset{0}{\cancel{\cos\phi_0}} - \omega_d C \overset{1}{\cancel{\sin(\phi_0)}}$$

$$\frac{I}{m} = \omega_d C \qquad C = \frac{I}{\omega_d m}$$

$$x(t) = \frac{I}{m\omega_d} e^{-\zeta\omega_n t}\sin(\omega_d t)$$

$$x(t) = I\, y_{under}(t)$$

$$\Big(= \frac{1}{m\omega_d} e^{-\zeta\omega_n t}\cos(\omega_d t + \phi_0)$$

"Impulse Response Function"

Depend on ζ could also be critical / overdamping

$$y_{crit}(t) = \frac{t}{m} e^{-\zeta \omega_n t}$$

$$y_{over}(t) = \frac{1}{2m\omega_n\sqrt{\zeta^2-1}}\left(e^{\omega_n\left(-\zeta+\sqrt{\zeta^2-1}\right)t} - e^{\omega_n\left(-\zeta-\sqrt{\zeta^2-1}\right)t}\right)$$

What if impulse is applied @ $t=\tau$

What is position $x(t)$

$$x(t) = I\, g(t-\tau)\cdot \underbrace{u(t-\tau)}_{\text{unit step}}$$

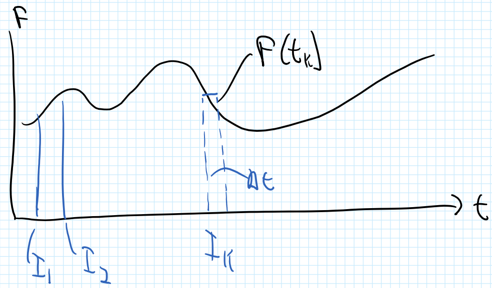
$$m\ddot{x} + c\dot{x} + kx = I\, \overbrace{\delta}^{\text{delta dirac function}}(t-\tau)$$

More than one impulse

Use linear superposition

$$m\ddot{x} + c\dot{x} + kx = I_1\,\delta(t-\tau_1) + I_2\,\delta(t-\tau_2)$$

$$x = I_1\, g_{under}(t-\tau_1)\, u(t-\tau_1) + I_2\, g_{under}(t-\tau_2)\, u(t-\tau_2)$$



The k^{th} impulse $\quad I_k = \int_{k\Delta t}^{(k+1)\Delta t} F(\tau)\, d\tau$

$$x_k(t) = I_k\, g(t - t_k)\, u(t - t_k)$$

$$F(\tau) = \sum_{k=1}^{n} I_k\, \delta(\tau - \tau_k)$$

$$x(t) = \sum x_k(t) = \sum I_k\, g(t - \tau_k)\, u(t - \tau_k)$$

take $\Delta\tau \to 0$ and integrate over τ

Insert integral definition of impulse $I = \int F\, dt$

$$x(t) = \int_0^T F(\tau)\, g(t - \tau)\, d\tau$$

Convolution Integral

Eg. Underdamped System

$$x(t) = \frac{1}{m\omega_d} \int_0^t F(\tau)\, e^{-\zeta\omega_n (t-\tau)} \sin(\omega_d (t - \tau))\, d\tau$$

Assumes initial impulse is at $t = 0$ and $\dot{x}(0) = 0$

1. What if we don't want $\dot{x}(0) = 0$

   Add an initiating impulse at $t = 0$

$$I = \Delta p = m\Delta v \to \Delta v = \frac{I}{m_{eq}}$$

2. What if you need a non-zero initial position

   i.e. don't start at equilibrium

Define a new variable

$$y \equiv x - x(0)$$

$$m\ddot{y} + c\dot{y} + ky = -kx(0) + F(t)$$

$$y(t) = \int_0^t \left[-kx(0) + F(\tau)\right] g(t-\tau)\, d\tau$$

$$y = x - x(0)$$

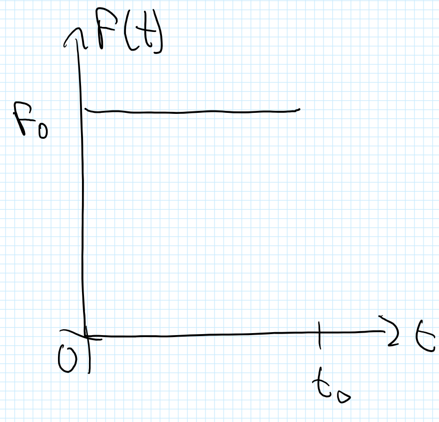
$$x = y + x(0)$$

General Underdamped Solution As

$$\boxed{\begin{aligned} x(t) = {} & x(0)\, e^{-\zeta\omega_n t} \cos(\omega_d t) + \frac{\dot{x}(0) + \zeta\omega_n x_0}{\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t) \\ & + \frac{1}{m\omega_d} \int_0^t F(\tau)\, e^{-\zeta\omega_n (t-\tau)} \sin\left(\omega_d (t-\tau)\right) d\tau \end{aligned}}$$

completely general (assuming underdamping)

Example 1



How would an undamped system respond to this force

Set $\zeta = 0$ in the expression above

Assume system starts at rest / equilibrium

$$x(t) = \frac{1}{m\omega_n} \int^t F(\tau) \sin\left(\omega_n (t-\tau)\right) d\tau$$

$$x(t) = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin(\omega_n(t-\tau))\, d\tau$$

$$F(t) \begin{cases} F_0, & 0 \le t \le t_0 \\ 0, & t > t_0 \end{cases}$$

For $0 \le t \le t_0$

$$x(t) = \frac{F_0}{m\omega_n} \int_0^t \sin(\omega_n(t-\tau))\, d\tau$$

$$= \frac{F_0}{m\omega_n^2} \left(1 - \cos(\omega_n t)\right)$$

$$= \frac{F_0}{k} \left(1 - \cos(\omega_n t)\right)$$

For $t > t_0$

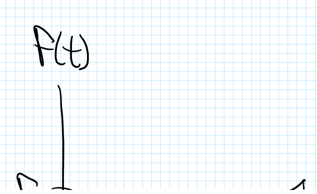
$$x(t) = \frac{F_0}{m\omega_n} \int_0^{t_0} \sin(\omega_n(t-\tau))\, d\tau$$

$$= \frac{F_0}{m\omega_n} \left(\frac{1}{\omega_n} \cos(\omega_n(t-\tau))\right)\Bigg|_{\tau=0}^{t_0}$$

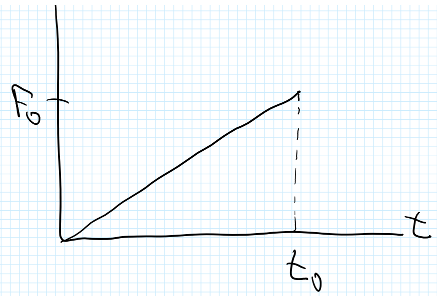
$$= \frac{F_0}{k} \left[\cos(\omega_n(t-t_0)) - \cos(\omega_n t)\right]$$

Example 2

Undamped, $x(0) = \dot{x}(0) = 0$



$$F(t) = \begin{cases} F_0 \dfrac{\tau}{t_0} & 0 \le t \le t_0 \\ 0 & t_0 \le t \end{cases}$$



$$F(t) = \begin{cases} F_0 \dfrac{t}{t_0} & \\ 0 & t_0 \le t \end{cases}$$

For $0 \le t \le t_0$

$$x(t) = \frac{F_0}{m\omega_n t_0} \int_0^t \tau \sin[\omega_n(t-\tau)]\, d\tau$$

$$= \frac{F_0}{k t_0} \left[t - \frac{1}{\omega_n} \sin(\omega_n t) \right]$$

For $t > t_0$

$$x(t) = \frac{F_0}{m\omega_n t_0} \int_0^{t_0} \tau \sin[\omega_n(t-\tau)]\, d\tau$$

$$x(t) = \frac{F}{k t_0} \left[\frac{1}{\omega_n} \sin(\omega_n(t-t_0)) + t_0 \cos[\omega_n(t-t_0)] - \frac{1}{\omega_n} \sin(\omega_n t) \right]$$

Numerical Methods

$$m\ddot{x} + c\dot{x} + Kx = f(t)$$

$$x_1(t) = x(t)$$

$$x_2(t) = \dot{x}(t) = \dot{x}_1(t)$$

$$m\dot{x}_2 = -cx_2 - Kx_1 + F(t)$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{c}{m}x_2 - \frac{K}{m}x_1 + \frac{1}{m}f(x)$$

$$\dot{\vec{x}} = F(\vec{x}, t)$$

$$\vec{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\vec{F}(\vec{x}, t) = \begin{Bmatrix} x_2 \\ -\frac{K}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}f(t) \end{Bmatrix}$$

$$x_{t=i+1} = x_{t=i} + \Delta x_{t=i}$$

Taylor Expansion

$$x(t+\Delta t) = x(t) + \dot{x}\Delta t + \ddot{x}\frac{\Delta t^2}{2} + \dots$$

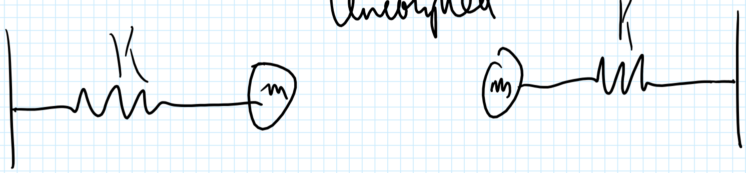
Take taylor expansion w/ some power K, w/ $(\Delta t)^K$

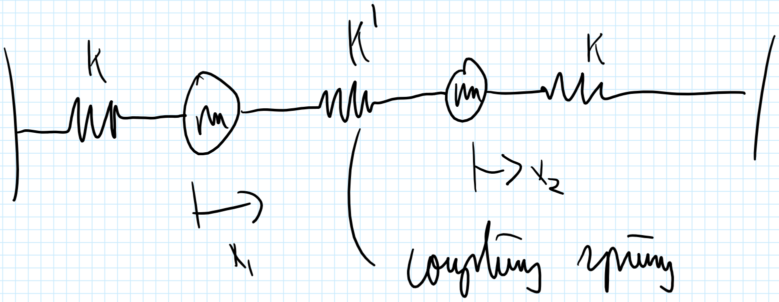
Initial Conditions

What about systems w/ multiple degrees of freedom

2 O.O.P

Uncoupled





I) $\sum F_1 = m\ddot{x}_1 = -Kx_1 - K'(x_1 - x_2)$

II) $m\ddot{x}_2 + Kx_2 + K'(x_2 - x_1) = 0$

$(\mathrm{I}) + (\mathrm{II}) \Rightarrow \quad m(\ddot{x}_1 + \ddot{x}_2) + K(x_1 + x_2) = 0$

$(\mathrm{I}) - (\mathrm{II}) \Rightarrow \quad m(\ddot{x}_1 - \ddot{x}_2) + K(x_1 - x_2) + 2K'(x_1 - x_2) = 0$

$\Psi_1 = x_1 + x_2$

$\Psi_2 = x_1 - x_2$

I and II
$$\left\{ \begin{array}{l} m\Psi_1 + K\Psi_1 = 0 \\ m\Psi_2 + (K + 2K')\Psi_2 = 0 \end{array} \right\}$$
– have solutions (Harmonics) for Ψ_1, Ψ_2

$$x_1(t) = \frac{1}{2}(\psi_1 + \psi_2) \implies x_1(t) = \frac{1}{2}\left[A_1 \cos(\omega_1 t + \theta_1) + A_2 \cos(\omega_2 t + \theta_2)\right]$$

$$x_2 = \frac{1}{2}(\psi_1 - \psi_2) = \frac{1}{2}\left[A_1 \cos(\omega_1 t + \theta_1) - A \cos(\omega_2 t + \theta_2)\right]$$

Leads to different modes

Lecture 20

Wednesday, March 31, 2021 3:05 PM

$t = 0$

$$\Psi_i = A \cos(\omega_i t + \theta_i)$$

$$x_1(0) = a \qquad x_2(0) = 0$$

$$\dot{x}_1(0) = 0 \qquad \dot{x}_2(0) = 0$$

$$\Psi_1 = x_1 + x_2$$

$$\Psi_1 = x_1 - x_2$$

$$\dot{\Psi}_1(0) = \dot{x}_1(0) + \dot{x}_2(0) = 0 = -A_1 \omega_1 \sin\theta_1 \Rightarrow \theta_1 = 0$$

$$\dot{\Psi}_2(0) = \dot{x}_1(0) - \dot{x}_2(0) = 0 = -A_2 \omega_2 \sin\theta_2 \Rightarrow \theta_2 = 0$$

$$\Psi_1(0) = a = A_1 \cos\theta_1 \Rightarrow A_1 = a$$

$$\Psi_2(0) = a = A_2 \cos\theta_2 \Rightarrow A_2 = a$$

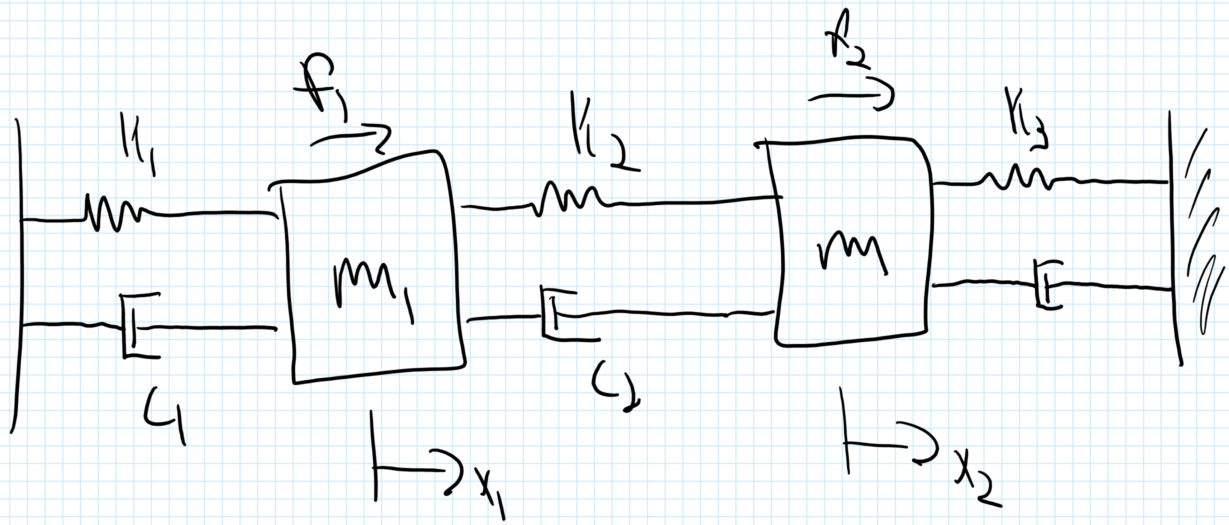
$$\psi_1(t) = a\cos(\omega_1 t)$$

$$\psi_2(t) = a\cos(\omega_2 t)$$

$$x_1(t) = a\cos\left(\frac{\omega_2-\omega_1}{2}t\right)\cos\left(\frac{\omega_2+\omega_1}{2}t\right)$$

$$x_2(t) = a\sin\left(\frac{\omega_2-\omega_1}{2}t\right)\sin\left(\frac{\omega_2+\omega_1}{2}t\right)$$

General 2. D.O.F System



$$\sum F_1 = m_1\ddot{x}_1 = -c_1\dot{x}_1 - c_2(\dot{x}_1-\dot{x}_2) - k_1 x_1 - k_2(x_1-x_2) + f_1$$

$\hookrightarrow$ Coupled ODE's

Write in matrix form

$$[m]\,\underline{\ddot{x}} + [C]\,\underline{\dot{x}} + [K]\,\vec{x} = \vec{f}$$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad C = \begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2+c_3 \end{bmatrix}$$

$$K = \begin{bmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2+K_3 \end{bmatrix} \quad f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

Plug in S.H.M formulas

$\Rightarrow$ only true if det is 0

$$\det \begin{pmatrix} -m_1\omega^2 + 2K & -K \\ -K & -m_2\omega^2 + 2K \end{pmatrix} = 0$$

$$\det \begin{pmatrix} -m_1\omega^2+2k & -k \\ -k & -m_2\omega^2+2k \end{pmatrix} = 0$$

↳ char eq for ω^2

$$\omega_1^2, \omega_2^2 = \frac{2k(m_1+m_2) \pm \sqrt{4k^2(m_1+m_2)^2 - 12k^2 m_1 m_2}}{2m_1 m_2}$$

check $m_1 = m_2 = m$ gives what we had before

These are eigen values, find eigen vectors

$\Rightarrow$ 1st eigen value

x_1 and x_2 have same amplitude and phase it

oscillating @ ω_1 ↳ symmetric mode

2nd eigen value

↳ anti-symmetric mode (beating)

Modal Vectors

- use amplitude ratios
- define solutions in vector form

What I.C. should we choose to generate motion of a certain mode

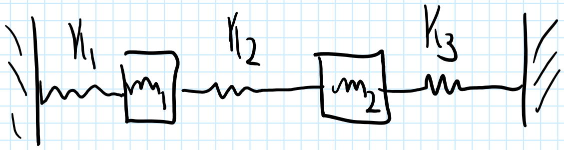
n^{th} normal mode will be excited iff

$\dot{x}_1(t=0) = \dot{x}_2(t=0) = 0$

$x_1(t=0) = A_1^{(n)}$

$x_2(t=0) = A_2^{(n)} = r_n A_1^{(n)}$

In general, motion can be written as linear combo of normal modes



$k_1 = 30 \frac{N}{m}$, $k_2 = 5 N/m$, $k_3 = 0 N/m$

$m_1 = 10$ kg, $m_2 = 1$ kg

IC's $x_1(0) = 1$, $\dot{x}_1(0) = 0 = x_2(0) = \dot{x}_2(0)$

$$\begin{bmatrix} -m_1\omega^2+k_1+k_2 & -k_2 \\ -k_2 & -m_2\omega^2+k_2+k_3 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

det = 0

$$\Rightarrow \quad \omega_1 = 1.58 \qquad \omega_2 = 2.45$$

Plug ω_1^2 into matrix

$$A_2^{(1)} = 2A_1^{(1)}$$

Plug ω_2^2 into matrix

$$A_2^{(2)} = -5A_1^{(2)}$$

$$x_1(t) = A_1^{(1)} \cos(\underset{\downarrow \omega_1}{1.58}\, t+\phi_1) + A_1^{(2)} \cos(\underset{\downarrow \omega_2}{2.45}+\phi_2)$$

$$x_2(t) = A_2^{(1)} \cos(1.58\, t+\phi_1) + A_2^{(2)} \cos(2.45+\phi_2)$$

Final Step, apply IC to get 4 unknowns

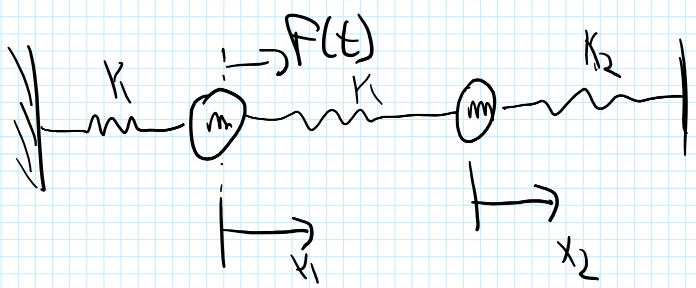
$$A_1^{(1)},\ A_1^{(2)},\ \phi_1,\ \phi_2$$

$\Rightarrow$ $x_1(t) = \ldots$

$x_2(t) = \ldots$

2. D.O.F in torsion

Forced Undamped 2 D-O-F systems



$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_0 \cos(\omega t) \\ 0 \end{bmatrix}$$

Just as before, separate in particular and homogeneous

We already know homogeneous solution

$$x_p(t) = \begin{Bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{Bmatrix} \cos(\omega t)$$

Plug into ODE and solve for $\bar{X}_1$ and $\bar{X}_2$

Two Resonance Frequencies

Forced undamped systems 2 D.O.F.

- Particular and Homogeneous solutions

assume particular solution harmonic w/ same freq as force

- leads to poles @ $\omega = \omega_1$, $\omega = \omega_2$

  "resonance frequencies"

Amplitudes X_1, X_2 will go to ∞

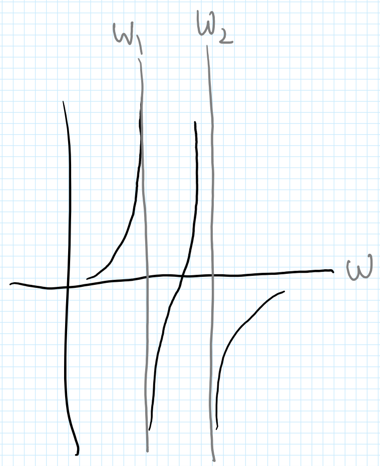
- analogous to 1 D.O.F. systems

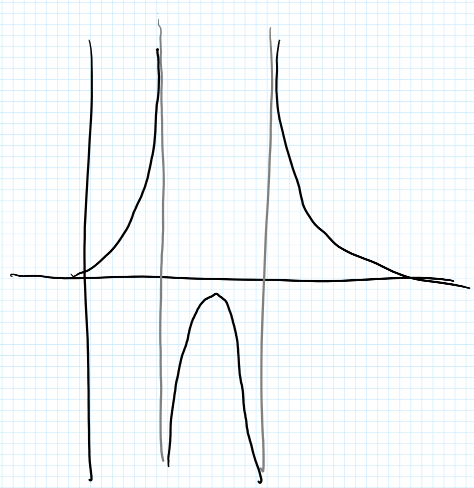
Key Features

$\omega < \omega_1$ Symmetric mode (both masses are in phase w/ force)

$\omega_1 < \omega < \omega_2$ 1. Symmetric mode, out of phase w/ driving force

2. as $\omega \to \omega_2$ Non-symmetric





Example: Vibration Absorber

$$K_1 = \frac{16 a t^3}{\ell^3} E$$

Equations of motion

$$m_1 \ddot{x}_1 + K_1 x_1 + K_2 (x_1 - x_2) = F_0 \cos(\omega t)$$

$$m_2 \ddot{x}_2 + K(x_2 - x_1) = 0$$

Assume Solutions of Harmonic Form

$$x_1(t) = X_1 \cos \omega t$$

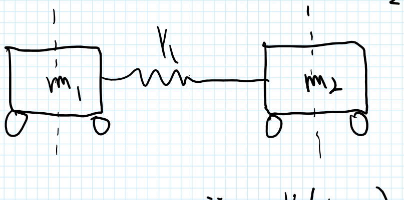
$$x_2(t) = X_2 \cos \omega t$$

Plug into ODE's

$$\left[-m_1 \omega^2 X_1 + K_1 X_1 + K_2 (X_1 - X_2)\right] \cancel{\cos \omega t} = F_0 \cancel{\cos \omega t}$$

$$-m_2 \omega^2 X_2 + K_2 (X_2 - X_1) = 0$$

Unrestrained or "semi-Definite" systems



$$m_1 \ddot{x}_1 + K(x_1 - x_2) = 0$$

$$m_2 \ddot{x}_2 + K(x_2 - x_1) = 0$$

Trial Solutions

$$x_1(t) = A_1 \cos(\omega t + \phi_1)$$

$$x_2(t) = A_2 \cos(\omega t + \phi_2)$$

Plug into ODE

$$-m_1\omega^2 A_1 + K(A_1 - A_2) = 0$$

$$\Rightarrow (-m_1\omega^2 + K)A_1 - KA_2 = 0$$

$$-m_2\omega^2 A_2 + K(A_2 - A_1) = 0$$

$$-KA_1 + (-m_2\omega^2 + K)A_2$$

$$\begin{bmatrix} -m_1\omega^2 + K & -K \\ -K & -m_2\omega^2 + K \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

set det $= 0$ to find non trivial solutions

$$\Rightarrow \quad \underbrace{\omega^2}_{0} \underbrace{\left[m_1 m_2 \omega^2 - K(m_1 + m_2)\right]}_{\text{or } 0} = 0$$

a) $\omega^2 = 0 \quad \Rightarrow \quad \omega_1 = 0$

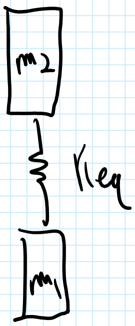
b) $m_1 m_2 \omega^2 - K(m_1 + m_2) = 0$

$$\omega_2 = \sqrt{\frac{K(m_1 + m_2)}{m_1 \cdot m_2}}$$

Lecture 23

Wednesday, April 14, 2021 3:15 PM

Rocket Example



E, A, L

$$K_{eq} = \frac{EA}{L}$$

Unrestrained System

$$\omega_1 = 0 \text{ rad/s}$$

$$\omega_2 = \sqrt{\frac{K_{eq}}{m_{red}}} = 4200 \text{ rad/s}$$

Plug in ω_2 to problem to find mode shape

$$\begin{bmatrix} -m_1\omega_2^2 + K & -K \\ -K & -m_2\omega_2^2 + K \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad A_2 = -0.2\, A_1$$

$$r_2 = -0.2$$

$$x_1(t) = \underbrace{x_{traj}(t)}_{\text{comes from mode 1 but since } \omega_1 = 0 \text{ this is just a traj of C.O.M.}} + A_1^{(2)} \cos(\omega_2 t + \phi_2)$$

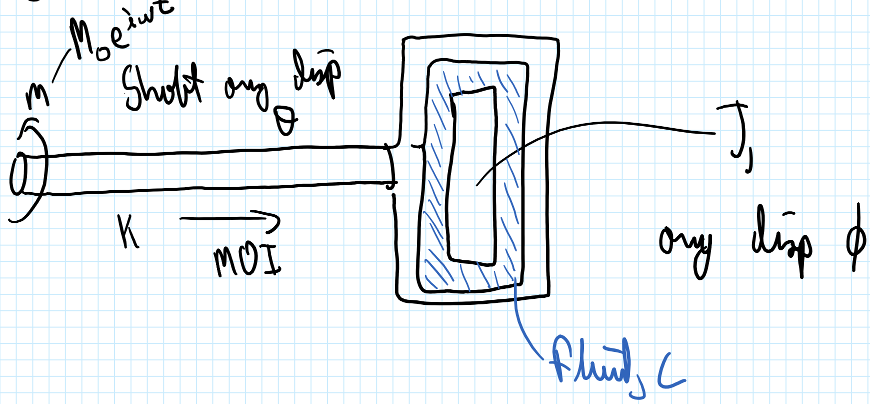
$$x_2(t) = x_{traj}(t) + r_2 A_1^{(2)} \cos(\omega_2 t + \phi_2)$$

Notice, we do know something about I.C.

(Static Thrust Force)

$$T = -Kx \Rightarrow x_2(0) - x_1(0) = 2.5 \cdot 10^{-5}$$

Damping of Systems w/ 2 D.O.F.



$$[m]\ddot{\vec{x}} + [c]\dot{\vec{x}} + [k]\vec{x} = \vec{F}$$

$$J\ddot{\theta} = -k\theta - c(\dot{\theta} - \dot{\phi}) + M_0 e^{i\omega t}$$

$$J\ddot{\theta} - c(\dot{\theta} - \dot{\phi}) + k\theta = M_0 e^{i\omega t}$$

$$J_1 \ddot{\phi} = c(\dot{\theta} - \dot{\phi})$$

$$J_1 \ddot{\phi} - c(\dot{\theta} - \dot{\phi}) = 0$$

$$\begin{bmatrix} J & 0 \\ 0 & J_1 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} + \begin{bmatrix} C & -C \\ -C & C \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \theta \\ \phi \end{bmatrix} = \begin{bmatrix} M_0 e^{i\omega t} \\ 0 \end{bmatrix}$$

Trial Solution

$$\theta = \theta_0 e^{i\omega t}$$

$$\phi = \phi_0 e^{i\omega t}$$

$$\Rightarrow \quad \left(-J\omega^2 + K + iC\omega\right)\theta_0 - iC\omega\phi_0 = M_0$$

$$\left(iC\omega - \omega^2 J_1\right)\phi - iC\omega\,\theta_0 = 0$$

$$\begin{bmatrix} -J\omega^2 + K + iC\omega & -iC\omega \\ -iC\omega & -\omega^2 J_1 + iC\omega \end{bmatrix}\begin{bmatrix} \theta_0 \\ \phi_0 \end{bmatrix} = \begin{bmatrix} M_0 \\ 0 \end{bmatrix}$$

Invert Matrix to get $\begin{bmatrix} \theta_0 \\ \phi_0 \end{bmatrix}$

$$\Rightarrow \quad \frac{\theta_0}{M_0} = \frac{\omega^2 J_1 - iC\omega}{\omega^2 J_1\left(K - J\omega^2\right) + iC\omega\left[\left(J + J_1\right)\omega^2 - K\right]}$$

$$\left|\frac{K\theta_0}{M_0}\right|$$

This is an unrestrained system

- modes are not directly coupled

  ($\omega_1 = 0$)

- If $\zeta = 0$, resonance occurs for $\omega = \omega_n$

- As $\zeta \to \infty$, resonance @ $\omega \approx \sqrt{\frac{K}{J+J_1}} = \omega_n \sqrt{\frac{1}{1+\mu}}$

How do we go to higher D.O.F

Inertia Matrix [m]

- usually diagonal matrix (uncoupled)

Damping and stiffness matrix

K_{ii} = (total stiffness of all ideal springs connected to m_i)

Analogous for C_{ii}

$K_{ij} = -$ (Total spring stiffness between mass i and j)

Analogous for C_{ij}

Lecture 24

Monday, April 19, 2021 4:13 PM

Phil didn't pay attention :/

Multiple D.O.F.

still write $[M]\ddot{x} + [C]\dot{x} + [K]x = F$

To determine complex eq. of motion in complicated systems it is helpful to use Lagranges Method

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i^{(n)} \qquad i = 1, 2, 3, \ldots n$$

For conservative system

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0 \qquad i = 1, 2, 3, \ldots, n$$

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0$$

eigen value problem

$$\left(-\omega^2 [m] + [k]\right) A = 0$$

Find modal vectors $[K] A^{(i)} = \omega_i^2 [M] A^{(i)}$

Define a matrix w/ all modal vectors

$$P \equiv \left[A^{(1)} \,\middle|\, A^{(2)} \,\middle|\, \ldots \,\middle|\, A^{(n)}\right]$$

Because of orthogonality we know

$$P^T [m] P = \begin{bmatrix} A^{(1)T} [m] A^{(1)} & & \\ & \ddots & \\ & & A^{(n)T} [m] A^{(n)} \end{bmatrix}$$

Same thing happens for $P^T [K] P$

$$\Rightarrow \quad \omega_i^2 = \frac{A^{(i)T} [m] A^{(i)}}{A^{(i)T} [K] A^{(i)}}$$ is just the ratio

of the diagonal elements

Equations of motion become

$$P^T [m] P \ddot{y} + P^T [K] P y = 0$$

What about forcing?

$$P^T [m] P \ddot{y} + P^T [K] P y = P^T F \equiv \hat{F}$$

$$\underbrace{\Phi^T [M] \Phi \ddot{y}} + \underbrace{\Phi^T [K] \Phi y} = \Phi^T F = \hat{F}$$

diagonalized, therefore uncoupled

Allows us to solve these independently

$$m_i \ddot{y}_i + k_i y_i = \hat{F}_i$$

Solve n forced "single d.o.f." problems

$$y_i(t) = y_{i,0} \cos(\omega_i t) + \frac{\dot{y}_{i,0}}{\omega_i} \sin(\omega_i t) + \int_0^t \hat{F}_i(\tau)\, g(t-\tau)\, d\tau$$

Lecture 26

Monday, April 26, 2021 3:02 PM

Diagonalize

→ normal coordinates

$$y = P^{-1} x$$

Take an impulse as an instantaneous change in velocity

Eg I_1 can set I.C.

$$\dot{x}_2(t=0) = \frac{I}{m_2} = 4 \text{ m/s}$$